

Universal Algebra Exercises - Sheet 3

If you have any questions about the exercises, feel free to email me at vojtech.david@matfyz.cuni.cz or ask during the practicals.

Exercise 1. Show that every complete lattice is bounded.

Exercise 2. Find examples of lattices L that contain a sublattice S such that

- (i) L is complete but S is not complete,
- (ii) L is not complete but S is complete,
- (iii) both L and S are complete lattices but S is not a complete sublattice.

Exercise 3. Let L be a complete lattice and $a, b \in L$ two compact elements.

- (i) Is $a \vee b$ compact?
- (ii) Is $a \wedge b$ compact?

Exercise 4. Let c be a closure operator on a set X . Prove that $\mathcal{L}_c$ is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$c(U \cup V) = c(U) \cup c(V).$$

Exercise 5. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset.$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

- (i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow \rightarrow}$ and $A^{\rightarrow \leftarrow}$ for both $A = \{\{1, 2\}, \{2, 3\}\}$ and $A = \{\{1, 2\}, \{2\}\}$. Compare the results.
- (ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.
- (iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow \leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}.$$

Exercise 6. Let c be a closure operator on a set X . Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$c(U) = U^{\rightarrow \leftarrow}$$

for all subsets $U \subseteq X$.

Exercise 7. A poset $(P, \leq)$ satisfies the ascending chain condition (ACC) if every infinite ascending chain stabilizes, i.e., for any sequence $(p_i)_{i \in \mathbb{N}}$ of elements of P such that

$$p_1 \leq p_2 \leq p_3 \leq \dots$$

there is $n \in \mathbb{N}$ such that for every $m \geq n$ we have $p_m = p_n$.

Let c be an algebraic closure operator on a set X . Show that the lattice $\mathcal{L}_c$ of closed sets satisfies ACC if and only if each closed set is finitely generated (i.e., if $A \subseteq X$ is closed, then there exists a finite $A_0 \subseteq A$ such that $A = c(A_0)$).