

Universal Algebra Exercises - Sheet 2

If you have any questions about the exercises, feel free to email me at vojtech.david@matfyz.cuni.cz or ask during the practicals.

Exercise 0. There are two natural ways to define the order relation on a given lattice $(L; \wedge, \vee)$:

$$x \leq y \iff x \wedge y = x \qquad \text{or} \qquad x \preceq y \iff x \vee y = y.$$

Show that these two orders are the same.

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{2.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{2.2}$$

Exercise 1. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 2. Show that in the definition of distributive lattices, the identities (2.1) and (2.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z.$$

Exercise 3. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

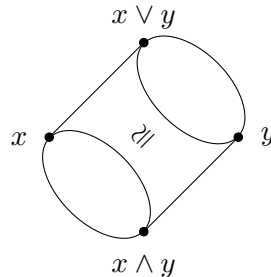
Exercise 4. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z),$$

$$x \leq z \implies x \vee (y \wedge z) \leq (x \vee y) \wedge z.$$

Exercise 5. Show that the lattice of normal subgroups of any group is modular. What about the lattice of all subgroups?¹

Exercise 6 (Diamond isomorphism theorem). Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.



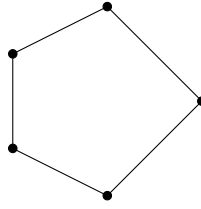
Exercise 7. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities

$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x).$$

Show that every lattice has a majority term.

¹A more general fact also holds for the lattices of submodules of modules. This is why we call this property *modular*.

Theorem (Dedekind). Prove that a lattice is modular if and only if it does not contain the following lattice (often called N_5) as a sublattice.



Theorem (Birkhoff). A modular lattice is distributive if and only if it does not contain the following lattice (often called M_3) as a sublattice.

