Universal Algebra Exercises - Homework 3

Exercise 1. Let R be a commutative ring, such that for every non-zero element a also $a^n \neq 0$ holds for all $n > 0$ (such a ring is called *reduced*).

(i) Show that for any non-zero $a \in R$ there is a prime ideal P_a with $a \notin P_a$. (Hint: Pick a P_a to be the greatest ideal that does not contain any power $a, a^2, a^3, \dots$)

(ii) Show that R is a subdirect product of integral domains.

Exercise 2. Let $\mathcal{V}$ be a variety of algebras $(A, \cdot, l, r)$ of type $(2, 1, 1)$ that satisfy the axioms

$$l(x \cdot y) = x, \quad r(x \cdot y) = y, \quad l(x) \cdot r(x) = x$$

(i) Show that every nontrivial member of $\mathcal{V}$ is infinite.

(ii) Show that, if $\mathbb{A}$ is generated by $\{a_1, a_2, \dots, a_n\}$, then it is already generated by $\{(a_1 \cdot a_2), \dots, a_n\}$

(iii) Let $\mathcal{F}_{\mathcal{V}}(n)$ be the free algebra in $\mathcal{V}$ with n generators. Show that $\mathcal{F}_{\mathcal{V}}(n) = \mathcal{F}_{\mathcal{V}}(m)$ for all positive integers n and m.

Exercise 3. Let $\mathbb{A}$ be the semigroup given by the following multiplication table.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

Prove the the variety generated by $\mathbb{A}$ is exactly the variety of all commutative semigroups satisfying $x^3 \approx x^4$.