Universal Algebra Exercises - Sheet 8

If $\mathcal{V}$ is a variety and X a set, let $F_{\mathcal{V}}(X)$ the free algebra in $\mathcal{V}$ on the set X.

Exercise 1. Let $\mathcal{V}$ be the variety of all algebras (A, f) where f is unary and $f^6 = f^2$. Determine and draw $F_{\mathcal{V}}(\{x\})$ and $F_{\mathcal{V}}(\{x, y\})$

Exercise 2. Let $\mathcal{S}$ be the variety of semigroups. Show that

$$F_{\mathcal{S}}(X) = (\{\text{nonempty words over } X\}, \circ)$$

where $\circ$ is concatenation of words (for example $(xyz) \circ (zy) = (xyzzy)$).

Exercise 3. Let $\mathcal{R}$ be the variety of semigroups satisfying

$$(x \cdot y) \cdot z \approx x \cdot z \quad \text{and} \quad x \cdot x \approx x$$

(i) Describe $F_{\mathcal{R}}(X)$ for any set X.

(ii) Find a natural homomorphism $F_{\mathcal{S}}(X) \to F_{\mathcal{R}}(X)$.

(iii) Generalize (ii) to free algebras of any varieties $\mathcal{W} \subseteq \mathcal{V}$.

Exercise 4. Let $\mathcal{V}$ be the variety of distributive lattices.

(i) Describe $F_{\mathcal{V}}(\{x\})$ and $F_{\mathcal{V}}(\{x, y\})$.

(ii) Find an upper bound on the size of $F_{\mathcal{V}}(\{x_1, \dots, x_n\})$

Remark. The question of whether $F_{\mathcal{V}}(X)$ is always finite for finite X is in general undecidable. It is even unknown if $F_{\mathcal{V}}(\{x, y\})$ is finite when $\mathcal{V}$ is the variety of groups with $(a^5 \approx 1)$. This is part of "Burnsides Problem".

Exercise 5. Let $\mathcal{V}$ be the variety of semilattices. Show that

$$F_{\mathcal{V}}(X) = (2^X \setminus \{\emptyset\}, \cup)$$