Universal Algebra Exercises - Sheet 12

Definition. An algebra $\mathbb{A}$ is called *congruence distributive* if its congruence lattice $\mathrm{Con}(\mathbb{A})$ is distributive. A variety $\mathcal{V}$ is called congruence distributive if it contains only congruence distributive algebras.

Theorem (Jónsson, 1967)**.** Let $\mathcal{V}$ be a variety and let $F = F_{\mathcal{V}}(\{x, y, z\})$ be the free algebra with three generators. The following are equivalent.

(i) $\mathcal{V}$ is congruence distributive

(ii) F is congruence distributive

(iii) there is an odd number n and ternary terms $J_0, \dots J_n$ that satisfy the following identities in $\mathcal{V}$:

- $J_0(x, x, y) \approx x$
- $J_n(x, y, y) \approx y$
- $J_i(x, y, x) \approx x$ for all i
- $J_i(x, x, y) \approx J_{i+1}(x, x, y)$ for even i
- $J_i(x, y, y) \approx J_{i+1}(x, y, y)$ for odd i

Exercise 1. Show that varieties with a majority term are congruence distributive. Can you do it without Jónssons theorem?

Exercise 2. Prove Jónssons theorem.

(i) Note that (i) $\Longrightarrow$ (ii) is trivial.

(ii) Let α, β and γ be the congruences in F generated by a single pair $(x, y), (y, z)$ and (x, z) respectively. Show that $(x, z) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

(iii) Conclude that there are elements $j_0, \dots j_n$ in F with

$$x(\alpha \wedge \gamma)j_0(\beta \wedge \gamma) \dots (\alpha \wedge \gamma)j_n(\beta \wedge \gamma)z$$

(iv) Use this to show (ii) $\Longrightarrow$ (iii).

(v) Try to show (iii) $\Longrightarrow$ (i).