Universal Algebra Exercises - Sheet 11

Exercise 51. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

*	0	1	2	3
0	1	2	1	0
1	0	3	2	3
2	1	0	1	0
3	2	3	2	1

Show that there is no function $f \in \mathrm{Clo}(\mathbb{A})$ satisfying the following.

(i) $f(3, 1, 3, 3, 3) = 0$

(ii) $f(1, 0, 2, 3, 2) = 0$ and $f(1, 0, 0, 3, 2) = 1$

Exercise 52. Consider a function $f : \{0, 1\}^n \to \{0, 1\}$ and recall the definition of its dual $f^d(x_1, \dots x_n) = \neg f(\neg x_1, \dots, \neg x_n)$.

(i) Show that $\mathrm{Pol}(\neq) = \{f \mid f^d = f\}$.

(ii) For any clone $\mathcal{C}$ on the set $\{0, 1\}$, show that $\mathcal{C}^d := \{f^d \mid f \in \mathcal{C}\}$ is also a clone.

(iii) Find a nice relational description of $\mathcal{C}^d$.

Exercise 53. For all numbers $n \geq 1$, define $\mathrm{OR}_n := \{0, 1\}^n \setminus \{(0, \dots, 0)\}$.

(i) Show that $\mathrm{Clo}(\{0, 1\}; \to) = \mathrm{Pol}(\mathrm{OR_n} \mid \mathrm{n} \geq 1)$

(ii) Show that $\mathrm{Clo}(\{0, 1\}; \to) \subsetneq \mathrm{Pol}(\mathrm{OR}_1, \dots, \mathrm{OR}_n)$ for any $n \geq 1$.

(iii) Show that $\mathrm{Clo}(\{0, 1\}; \to) \neq \mathrm{Pol}(R_1, \dots, R_n)$ for any finite set of relations $R_1, \dots, R_n$.