Universal Algebra Exercises - Sheet 10

Exercise 48. Let G be a group and let $\mathcal{A}$ be the variety of abelian groups. Let $\lambda_{\mathcal{A}}^{G}$ be the smallest congruence of G with an abelian quotient.

- Show that the congruence class of the identity element is the subgroup of G generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$.

$$1/\lambda_{\mathcal{A}}^{G} = \mathrm{Sg}_G([x, y] \mid x, y \in G)$$

- Show that the variety $\mathcal{A} \cdot \mathcal{A}$ is axiomatized by the group laws and $[[x, y], [z, w]] = 1$.

Exercise 49. Let $\mathcal{A}_n$ be the variety of abelian groups satisfying $x^n \approx 1$. Show that

- $\mathcal{A}_3 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{x^6 \approx 1, \quad [x^2, y^2] \approx 1, \quad [x, y]^3 \approx 1\}\Big)$
- $\mathcal{A}_2 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{(x^2y^2)^2 \approx 1\}\Big)$

Exercise 50. Let cRing_n be the variety of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ be the field of order 9. Show that $\mathrm{HSP}(\mathbb{F}_9)$ is axiomatized by the axioms of cRing_9 together with

$$x + x + x \approx 0.$$

Exercise 51. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

*	0	1	2	3
0	1	2	1	0
1	0	3	2	3
2	1	0	1	0
3	2	3	2	1

Show that there is no function $f \in \mathrm{Clo}(\mathbb{A})$ satisfying the following.

(i) $f(3, 1, 3, 3, 3) = 0$

(ii) $f(1, 0, 2, 3, 2) = 0$ and $f(1, 0, 0, 3, 2) = 1$