The Rise of Plurimorphisms: Algebraic Approach to Approximation

Libor Barto — Charles University
Silvia Butti — University of Oxford
Alexandr Kazda — Charles University
Caterina Viola — University of Catania
Stanislav Živný — University of Oxford

Dagstuhl, 19 May 2025

Funded by the European Union

European Research Council
Established by the European Commission

Funded by the European Union (project POCOCOP, ERC Synergy grant No. 101071674). Views and opinions expressed are however those of the author only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

INTRO

$\boxed{\text{Dream Theorem}}$ For P, P' $\quad * \iff P' \leq P$

computational problems

nice condition

poly-time reduction

$\boxed{\text{Real theorems}}$ For $P, P' \in$ Class $\quad * \implies P' \leq P$

We have
- such theorem for Class = fixed-template finite-domain CSPs
- $*$ weak enough to "explain" NP-hardness:

  $P \in$ Class NP-hard, $P' \in$ Class $\Rightarrow *$ is satisfied

We want
- make Class bigger ⟶ this talk
- make $*$ weaker

FIXED TEMPLATE CSPs

CSP $\xrightarrow{\text{optimization}}$ Valued CSPs $\xrightarrow{\text{approx.}}$ Valued PCSPs

CSP $\xrightarrow{\text{approximation}}$ Promise CSPs $\xrightarrow{\text{opt.}}$ Valued PCSPs

NEW

CSP

- 3-coloring
- 3LIN2

$x_1 + x_2 + x_3 = 1 \pmod 2$

Valued CSPs

- Max 3-coloring
- Max 3LIN2

Promise CSPs

- 3-vs-6-coloring

Valued PCSPs

- 0.9-vs-0.6 3LIN2
- 1-vs-0.1 Label Cover
- 0.9-vs-0.1 Unique Games

- finite domains
- opinion: remove ~~P~~ ~~Valued~~ CSP - it is the template which is more general not the problem

PCSP vs Valued PCSP

template

$\mathbb{A} \begin{cases} \bullet\ A \text{ finite domain} \\ \bullet\ \varphi^{\mathbb{A}}: A^{[n]} \to \{\text{true}, \text{false}\} \end{cases}$

$\mathbb{B} \begin{cases} \bullet\ B \\ \bullet\ \varphi^{\mathbb{B}}: B^{[n]} \to \{\text{true}, \text{false}\} \end{cases}$

$\mathbb{A} \begin{cases} \bullet\ A \\ \bullet\ \varphi^{\mathbb{A}}: A^{[n]} \to \mathbb{Q} \cup \{-\infty\} \end{cases}$

payoff function
valued relation

- $c \in \mathbb{Q}$ completeness

$\mathbb{B} \begin{cases} \bullet\ B \\ \bullet\ \varphi^{\mathbb{B}}: B^{[n]} \to \mathbb{Q} \cup \{-\infty\} \end{cases}$

- $s \in \mathbb{Q}$ soundness

PCSP($\mathbb{A}, \mathbb{B}$)

INPUT: conjunctive formula over X

e.g. $\Phi := \varphi(x,y) \wedge \varphi(z,y)$

YES: $\exists h: X \to A \quad \Phi^{\mathbb{A}}(h)$

NO: $\neg \exists h: X \to B \quad \Phi^{\mathbb{B}}(h)$

PCSP($\mathbb{A}, \mathbb{B}, c, s$)

payoff formula over X

weights ≥ 0

e.g. $\Phi := 0.3\,\varphi(x,y) + 0.8\,\varphi(z,y)$

$(w(\Phi) = 0.3 + 0.8 = 1.1)$

$\exists h: X \to A \quad \Phi^{\mathbb{A}}(h) \geq c \cdot w(\Phi)$

$\neg \exists h: X \to B \quad \Phi^{\mathbb{B}}(h) \geq s \cdot w(\Phi)$

EXAMPLE & NOTES

$\mathbb{A}$ $\begin{cases} \bullet\ A = \{0,1\} \\ \bullet\ \varphi^{\mathbb{A}}(x,y,z) = \begin{cases} 1 & \text{if } x+y+z=1 \pmod 2 \\ 0 & \text{else} \end{cases} \end{cases}$

- $c = 0.9$

$\mathbb{B}$ $\begin{cases} \bullet\ B = A,\ \varphi^{\mathbb{B}} = \varphi^{\mathbb{A}} \end{cases}$

- $s = 0.6$

PCSPs $(\mathbb{A}, \mathbb{B}, c, s)$

INPUT: (weighted) system of linear eq. mod 2

e.g.
$$x_1 + x_5 + x_7 = 1$$
$$x_2 + x_8 + x_{17} = 1$$
$$\dots$$

YES: $\exists h: X \to \{0,1\}$ satisfying
0.9-fraction of equations

NO: $\neg \exists h: X \to \{0,1\}$ satisfying
0.6-fraction of equations

Remarks

- more relation symbols φ
- multisorted (multiple domains)
- YES, NO should be disjoint $\to$ assumption
- Opinion: input is <u>not</u> the same sort of objects as the template
  (non-valued — valued; limiting $\times$ allowing; weights $\times$ payoffs)

Versions

- maximization vs minimization
- WLOG $c = s = 0$
- alternatively $w(\Phi) = 1$
  (instance.. probability distribution on constraints)
- all weights equal (less natural)
- do not fix c, s
  fix $\alpha := s/c$ (constant-factor approximation)

REDUCTION THEOREMS

Reduction theorem For $P, P' \in$ Class $\quad * \Rightarrow P' \leq P$

more abstract problems

Better reduction theorem For $P, P' \in$ Class find $abs(P), abs(P')$ so that

(1) $P \sim abs(P) \; (P' \sim abs(P'))$ $\quad$ (1.1) $P \leq abs(P)$ $\quad$ (1.2) $abs(P) \leq P$

(2) $* \Rightarrow abs(P') \leq abs(P)$

Done for CSPs & beyond

minion of polymorphisms

• "minor condition"
• version of Label Cover

• $P = CSP(\mathbb{A})$, $\mathcal{M} = Pol(\mathbb{A})$, $abs(P) = MC(\mathcal{M})$

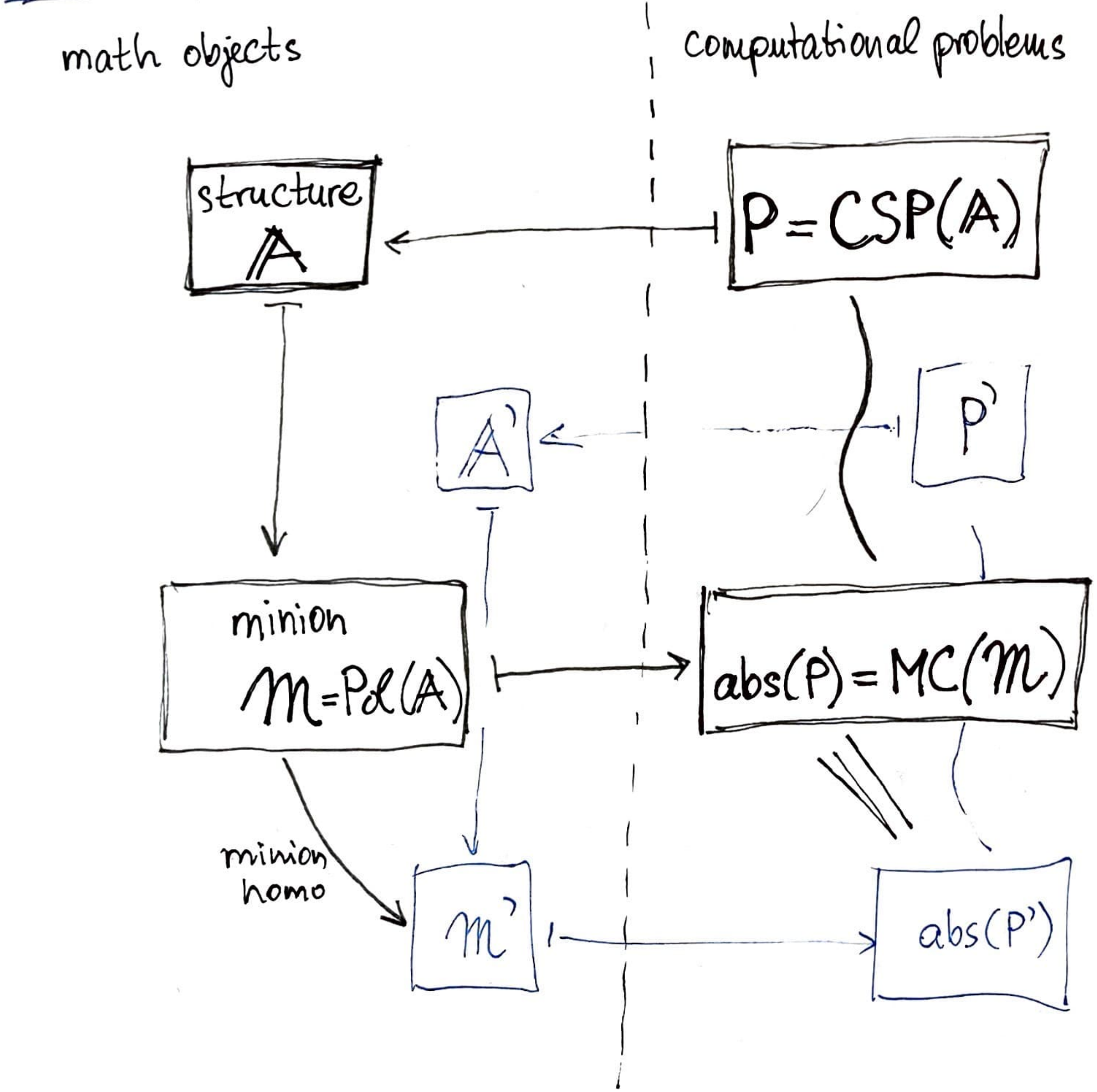
$P' = \ldots$

(1) $CSP(\mathbb{A}) \sim MC(\mathcal{M})$

(2) $\exists\, \mathcal{M} \xrightarrow{\text{minion homo}} \mathcal{M}' \Rightarrow MC(\mathcal{M}') \leq MC(\mathcal{M})$

Our work ditto for Valued PCSPs

PICTURE FOR CSPs



abstraction:

- simpler objects
- less parameters
- less data to desribe it
- smarter objects
- less elementary
- better

Our work

- phrases PCSP basics in this way including proofs
- generalizes it to Valued PCSPs

PRESERVATION

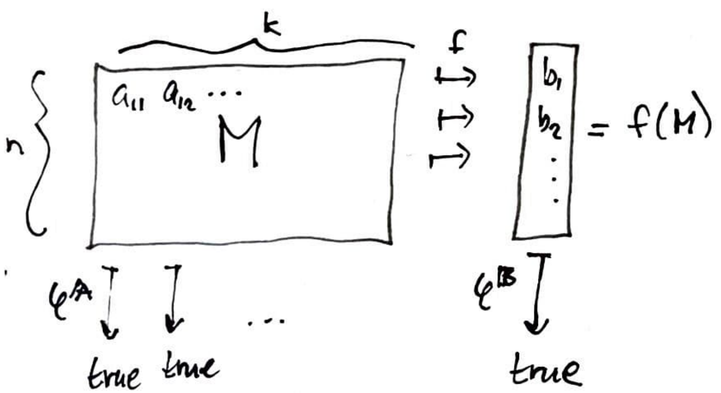
PCSP

- $(\mathbb{A}, \mathbb{B})$ PCSP template
  $\varphi^{\mathbb{A}}: A^{[n]} \longrightarrow \{\text{true}, \text{false}\}$
  $\varphi^{\mathbb{B}}: B^{[n]} \longrightarrow \{\text{true}, \text{false}\}$

- f k-ary function
  $f: A^{[k]} \longrightarrow B$

- $M \in A^{[n] \times [k]}$ matrix

- f preserves M if
  $\forall i \quad \varphi^{\mathbb{A}}(\text{col}_i M) \Rightarrow \varphi^{\mathbb{B}}(f(M))$



Valued PCSP

- $(\mathbb{A}, \mathbb{B}, c, s)$ Valued PCSP template
  $\varphi^{\mathbb{A}}: A^{[n]} \longrightarrow \mathbb{Q}$ (without $-\infty$)
  $\varphi^{\mathbb{B}}: B^{[n]} \longrightarrow \mathbb{Q}$

- $\Omega = (\Omega^{in}, \Omega^{out})$ k-ary weighting
  Ω^{in} probability distribution on $[k]$
  Ω^{out} probability distribution on $\{f;\ f: A^{[k]} \to B\}$

- $M \in A^{[n] \times [k]}$ matrix

- f preserves M if

$$\underset{i \sim \Omega^{in}}{\mathbb{E}} \varphi^{\mathbb{A}}(\text{col}_i M) \geq c \Rightarrow \underset{f \sim \Omega^{out}}{\mathbb{E}} \varphi^{\mathbb{B}}(f(M))$$

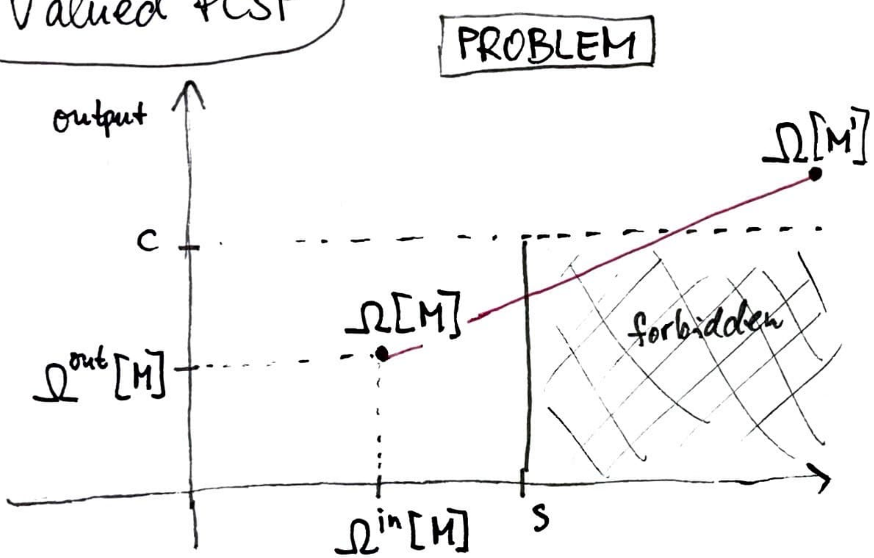
$$\underset{i \sim \Omega^{in}}{\mathbb{E}} \varphi^{\mathbb{A}}(\text{col}_i M) = \ddot{\Omega}^{in}[M] \qquad \underset{f \sim \Omega^{out}}{\mathbb{E}} \varphi^{\mathbb{B}}(f(M)) = \ddot{\Omega}^{out}[M]$$

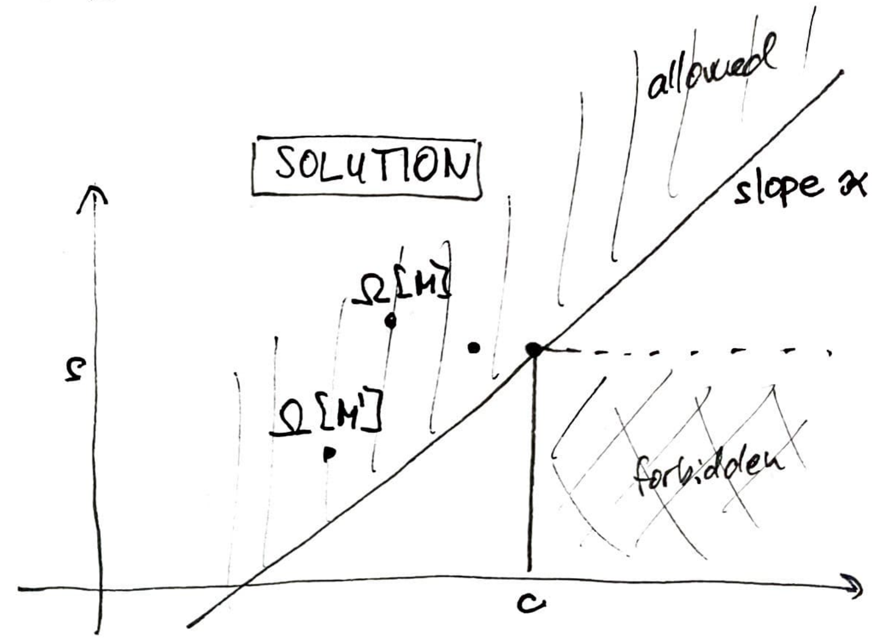
PLURIMORPHISMS

PCSP

- $\mathrm{Pol}(\mathbb{A},\mathbb{B}) = \{ f ; f$ preserves every $M \}$ (regarded as abstract minion)
- important: $\forall f \in \mathrm{Pol}(\mathbb{A},\mathbb{B})$ preserves the solution set of every Φ
- no longer true for Valued PCSPs

Valued PCSP





- For $\varkappa \geq 0$ Ω is $\varkappa$-polymorphism if $\Omega^{out}[M] - s \geq \varkappa(\Omega^{in}[M] - c)$
- Plurimorphism family $(\Omega_j)_{j \in J}$ s.t. $\exists \varkappa$ $\forall \Omega_j$ is a $\varkappa$-polymorphism
- $\leadsto$ valued minion homomorphisms, ...

MINOR CONDITION PROBLEM (Label Cover version)

parameters: $\mathcal{M}$ minion, $k \in \mathbb{N}$

INPUT (label cover instance)
- disjoint U, V
- D_x with $|D_x| \leq k$ $\forall x \in U \cup V$
- constraints of the form

$$\pi(u) = v \quad \left(\begin{array}{l} u \in U, v \in V \\ \pi : D_u \to D_v \end{array}\right)$$

YES: $\exists h : U \cup V \to \ldots$ $h(x) \in D_x$

for each $\pi(u) = v$ $\quad \pi(h(u)) = h(v)$

NO: $\neg \exists h : U \cup V \to \ldots$ $h(x) \in \mathcal{M}^{(D_x)}$

for each $\pi(u) = v$ $\quad \mathcal{M}^{(\pi)}(h(u)) = h(v)$

$\mathcal{M}$ minion, $\mathbb{M}$ valued minion over $\mathcal{M}$, $k \in \mathbb{N}$

- U, V
- D_x with $|D_x| \leq k$
- constraints $\pi(u) = v$
- $\alpha_u : D_u \to \mathbb{Q}$ for $\forall u \in U$ — input payoffs
  $\beta_u : \mathcal{M}^{(D_u)} \to \mathbb{Q}$ — output payoffs

$\mathcal{M}^{(D_u)}$ D_u-ary members

such that

$$\forall (\Omega_u)_{u \in U} \quad \sum_{u \in U} \Omega_u^{in}[\alpha_u] \geq 0 \Rightarrow \sum_{u \in U} \Omega_u^{out}[\beta_u] \geq 0$$

YES: $\exists h \ldots$ and $\sum_{u \in U} \alpha_u(h(u)) \geq 0$

NO: $\exists h \ldots$ and $\sum_{u \in U} \beta_u(h(u)) \geq 0$

SUMMARY

- Valued PCSP framework, includes
  - 0.9-vs-0.6 3LIN2, 1-vs-0.1 Label Cover, 0.9-vs-0.1 Unique Games
- Reduction theorem, based on MC
  - includes gadget reductions
  - explains $GLC(1, \varepsilon) \leq$ 0.9-vs-0.6 3LIN2 (TODO: explain Raghavendra)
- Harder part: concepts, easier part: proofs
- Main TODO: explain $3SAT \leq GLC(1, \varepsilon)$ (PCP theorem)

  good: this ~~[illegible]~~ task now makes sense

Thank You

BONUS: REDUCTIONS FROM GLC

$\mathrm{GLC}_{D,E}(1,\varepsilon) \leq \mathrm{PCSP}(\mathbb{A},\mathbb{B},c,s)$ under the following conditions is explained by the reduction theorem (remark: $\leq$ itself is standard + easy; this formulation is noteworthy + the explain part)

There exist
- $\Lambda_D : \mathcal{M}^{(D)} \to$ probability distr. on D $\Big($ $\mathcal{M} = \mathrm{PolFeas}(\mathbb{A},\mathbb{B})$ $\Big)$
  $\Lambda_E : \mathcal{M}^{(E)} \to$ — " — E
- $\forall \pi : D \to E$ a normalized Φ_π over $A^D \cup A^E$
  a linear nondecreasing $\gamma_\pi : \mathbb{R} \to \mathbb{R}$ with $\gamma_\pi(s) \geq \varepsilon$

such that $\forall \pi : D \to E$
- $\Phi_\pi^{\mathbb{A}}(\mathrm{proj}^D_d, \mathrm{proj}^E_{\pi(d)}) \geq c$ for every $d \in D$
- $\gamma_\pi\left(\Phi_\pi^{\mathbb{B}}(f_D, f_E)\right) \leq \underset{\substack{d \sim \Lambda_D(f_D) \\ e \sim \Lambda_E(f_E)}}{\mathbb{E}} \pi(d,e)$ for every $f_D \in \mathcal{M}^{(D)}$, $f_E \in \mathcal{M}^{(E)}$