Solving the $\square$

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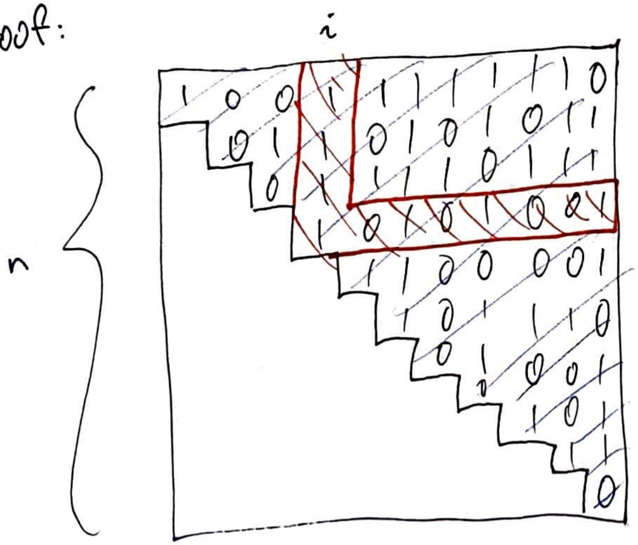
THE FIRST STEP

- vector relaxation over $\mathbb{Z}_2$
  - how to compute in P
- using the Fundamental Theorem on PCSPs in algorithmic way
- fun rounding

EVERY MATRIX IS A GRAM MATRIX

Theorem: Let M be an $n \times n$ ~~matrix~~ symmetric matrix over $\mathbb{Z}_2$. Then $\exists N \ \exists \vec{v_1}, \vec{v_2}, \dots, \vec{v_n} \in \mathbb{Z}_2^N$ such that $\forall i,j \ M_{ij} = \vec{v_i} \cdot \vec{v_j}$

Proof:



- $N :=$  $|N| = \frac{n \cdot (n+1)}{2}$
- $\vec{v_i} =$ , zeros elsewhere
- for $i \neq j$ $\vec{v_i} \cdot \vec{v_j} = M_{ij}$
- adjust diagonal entries so that $\vec{v_i} \cdot \vec{v_i} = M_{ii}$

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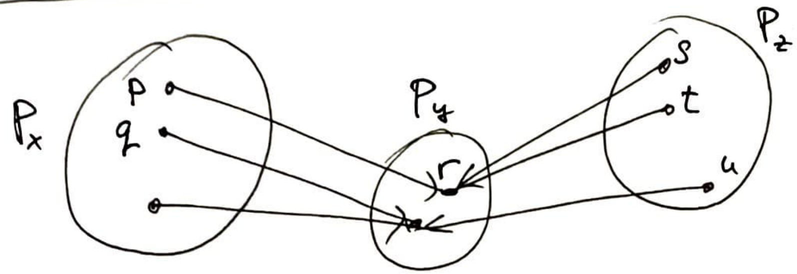
for other prime~~s~~ fields: simple adjustment

VECTOR RELAXATION OVER $\mathbb{Z}_2$



$P :=$ points

Vector $\mathbb{Z}_2$-solution: $\vec{v} : P \longrightarrow \mathbb{Z}_2^N$

- summing condition (e.g. $\vec{v}(s) + \vec{v}(t) = \vec{v}(r)$)
- $\forall x \quad \sum_{p \in P_x} \vec{v}(p) = \vec{i}$, weight$(\vec{i}) = 1$ $(= \vec{i} \cdot \vec{i})$
- $\forall x \quad \forall p \neq q \in P_x \quad \vec{v}(p) \cdot \vec{v}(q) = 0$

How to find it

- $R_1 :=$ $\mathbb{Z}_2$-polynomials with variables $p \in P$ of degree ≤ 1
  $N_1 := \mathrm{Span}\,\{ s+t-r, \dots, s+t+u=1 \}$
- Find symmetric bilinear form $\beta : R_1/N_1 \times R_1/N_1 \to \mathbb{Z}_2$ such that
  $p \neq q \in P_x \Rightarrow \beta(p/N_1, q/N_1) = 0$, $\beta(1/N_1, 1/N_1) = 1$ (!! must exist)
  $\beta(p/N_1, q/N_1) := \vec{v}(p) \cdot \vec{v}(q)$
- Use previous slide to find vectors

- + D. Zhuk: such a β is exactly a solution of 2nd level of $\mathbb{Z}_2$
- D. Zhuk: $\exists$ version for k-th level $\mathbb{Z}_m$ for any prime $m > 2$

SOLVING $\square$

D_4 = 8-element dihedral group

CSP over constraint language Γ

version 1 $\quad \Gamma$ = relations invariant under $\bullet$ = subgroups of D_4^n

version 2 $\quad \Gamma$ = ——"—— and $\{a\}$ for $a \in D_4$

version 3 $\quad \Gamma$ = cosets of subgroups of D_4 (= invariant under $xy^{-1}z$)

version 1 – trivial < version 2 (will show) < version 3 & (will actually solve)

how to do it

→ explore invariant relations, ... (see Zhuk's Singleton (BLP+AIP) paper for relational generators)

→ use the Fundamental Theorem

the first time applied algorithmically

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THE FUNDAMENTAL THEOREM

Given $\mathcal{M}$... minion, $S \in \mathbb{N}$ (Relaxed $LC_S(\mathcal{M})$) (a.k.a. minor condition problem [BBKO])

INPUT: Label Cover instance $\mathcal{D}$, domain sizes $\leq S$

YES: $\exists$ solution to $\mathcal{D}$

NO: $\neg\exists$ solution to $\mathcal{M}$-relaxed $\mathcal{D}$ ..... $D \xrightarrow{\alpha} E \rightsquigarrow \mathcal{M}^{(D)} \xrightarrow{\mathcal{M}^{(\alpha)}} \mathcal{M}^{(E)}$

$\boxed{\text{Thm}}$ $\mathbb{A} \to \mathbb{B}$ finite, $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$, S large enough. Then

$$\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \sim \text{Relaxed } LC_S(\mathcal{M})$$

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will show that for our Γ, $\mathcal{M} = \mathrm{Pol}(\Gamma)$ Relaxed $LC_\infty(\mathcal{M})$, even the search version, is in P:

~~given~~ Search version: <u>Given</u> Label Cover instance $\mathcal{D}$

<u>Find</u> solution to $\mathcal{M}$-relaxed $\mathcal{D}$

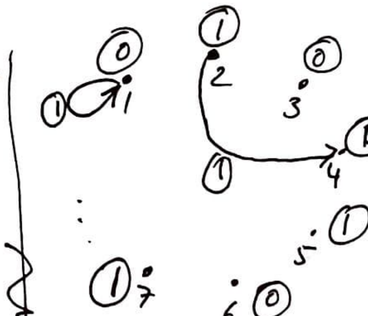
$\boxed{\mathcal{M}}$

$\mathcal{M} = \mathrm{Pol}(\Gamma) =$ idempotent term operations of D_4

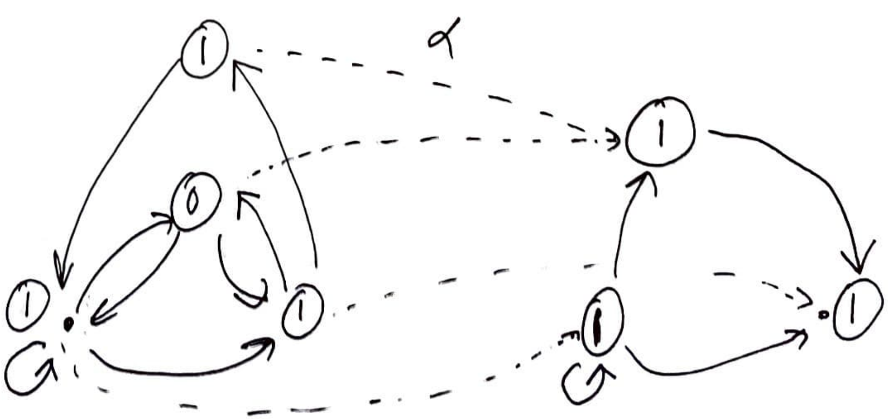
- $D_4 = (\mathbb{Z}_2^3, \cdot)$ $\quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \\ c+c' + ab' \end{pmatrix}$ ← all computations in $\mathbb{Z}_2$

more complex term operation $\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}, \dots \right) \mapsto \begin{pmatrix} a_2 + a_5 + a_7 \\ b_2 + b_5 + b_7 \\ c_2 + c_5 + c_7 + a_1 b_1 + a_2 b_4 + \dots \end{pmatrix}$

$$\mathcal{M}^{(D)} = \left\{ f : D \to \mathbb{Z}_2,\ g : D \times D \to \mathbb{Z}_2 \ ; \ \begin{array}{l} \sum f(d) = 1 \\ \sum\limits_{d,e \in D^2} g(d,e) = 0 \\ d \neq e \Rightarrow g(d,e) + g(e,d) = f(d) \cdot f(e) \end{array} \right\}$$



$\mathcal{M}^{(\alpha)}$ – summation



ROUNDING

For $v_1 v_2 \ldots v_N$, $w_1 w_2 \ldots w_N \in \mathbb{Z}_2^N$ let

- $\omega(v_1 \ldots v_N) = \sum v_i$ (the weight)
- $\gamma(v_1 \ldots v_N, w_1 \ldots w_N) = \sum_{i<j} v_i w_j$

$$\text{⑤} \quad \gamma(\vec{v}, \vec{w}) + \gamma(\vec{w}, \vec{v}) = \omega(\vec{v}) \cdot \omega(\vec{w}) + \vec{v} \cdot \vec{w}$$

- Take ~~diagram~~ Label Cover instance $\mathcal{D}$
- Take vector $\mathbb{Z}_2$-solution $\vec{v}: P \longrightarrow \mathbb{Z}_2^N$
- for every variable x, define $(f_x, g_x) \in \mathcal{M}^{(D_x)}$ by (~~where $p = xa$, $q = xb$~~)
  - $f_x(p) = \omega(\vec{v}(p))$
  - $a \neq b$ $\quad g_x(p, q) = \gamma(\vec{v}(p), \vec{v}(q))$
  - $g_x(p, p) = \gamma(\vec{v}(p), \vec{v}(p) - \vec{1})$