# Symmetry of Constraints

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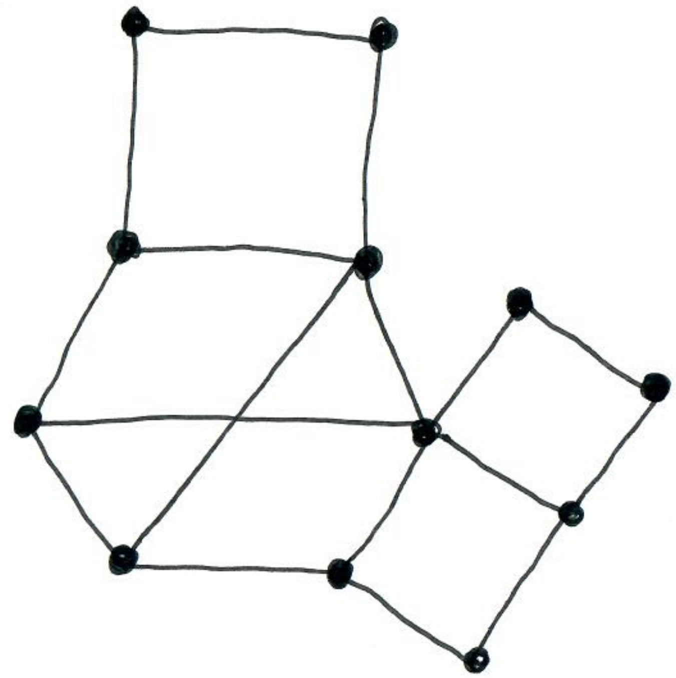
# Are these shapes symmetric?

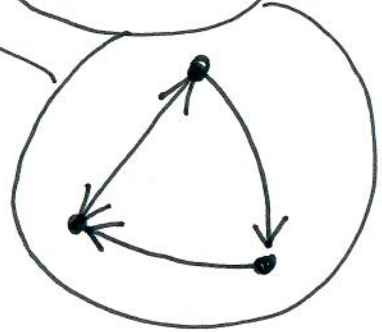
1

2

3



# graphs, digraphs, homomorphisms

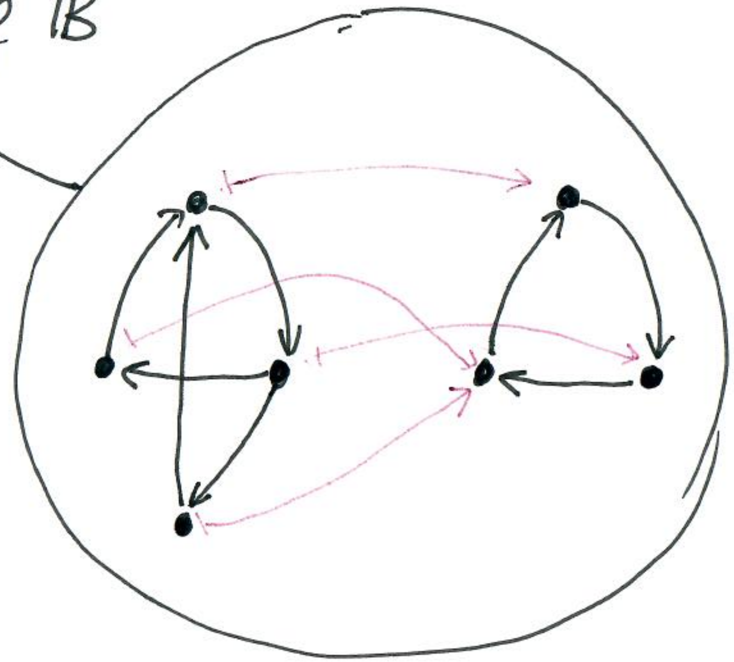


<u>graph</u> $\mathbb{A}$ : vertices, edges

<u>digraph</u> $\mathbb{A}$ : vertices, directed edges

<u>homomorphism</u> $\mathbb{A} \to \mathbb{B}$ :

mapping vertices of $\mathbb{A}$ → vertices of $\mathbb{B}$
that preserves edges



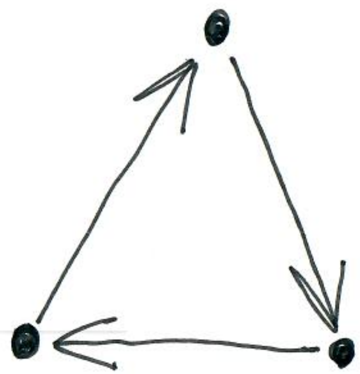
<u>endomorphism</u> of $\mathbb{A}$ : $\mathbb{A} \to \mathbb{A}$

<u>automorphism</u> of $\mathbb{A}$ : invertible $\mathbb{A} \to \mathbb{A}$

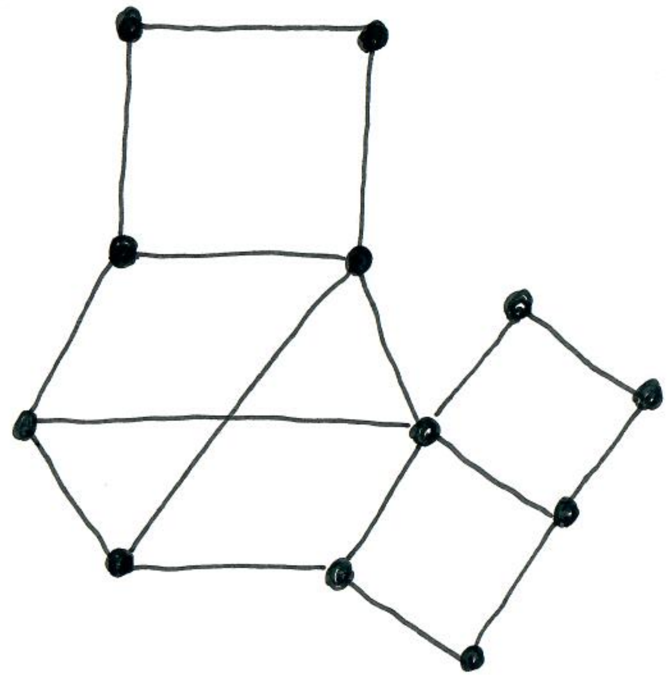
# Are these shapes symmetric?

1

2



3



# Outline

- CSP
- CSPs & Symmetries
- Analysis of symmetries

# CSPs

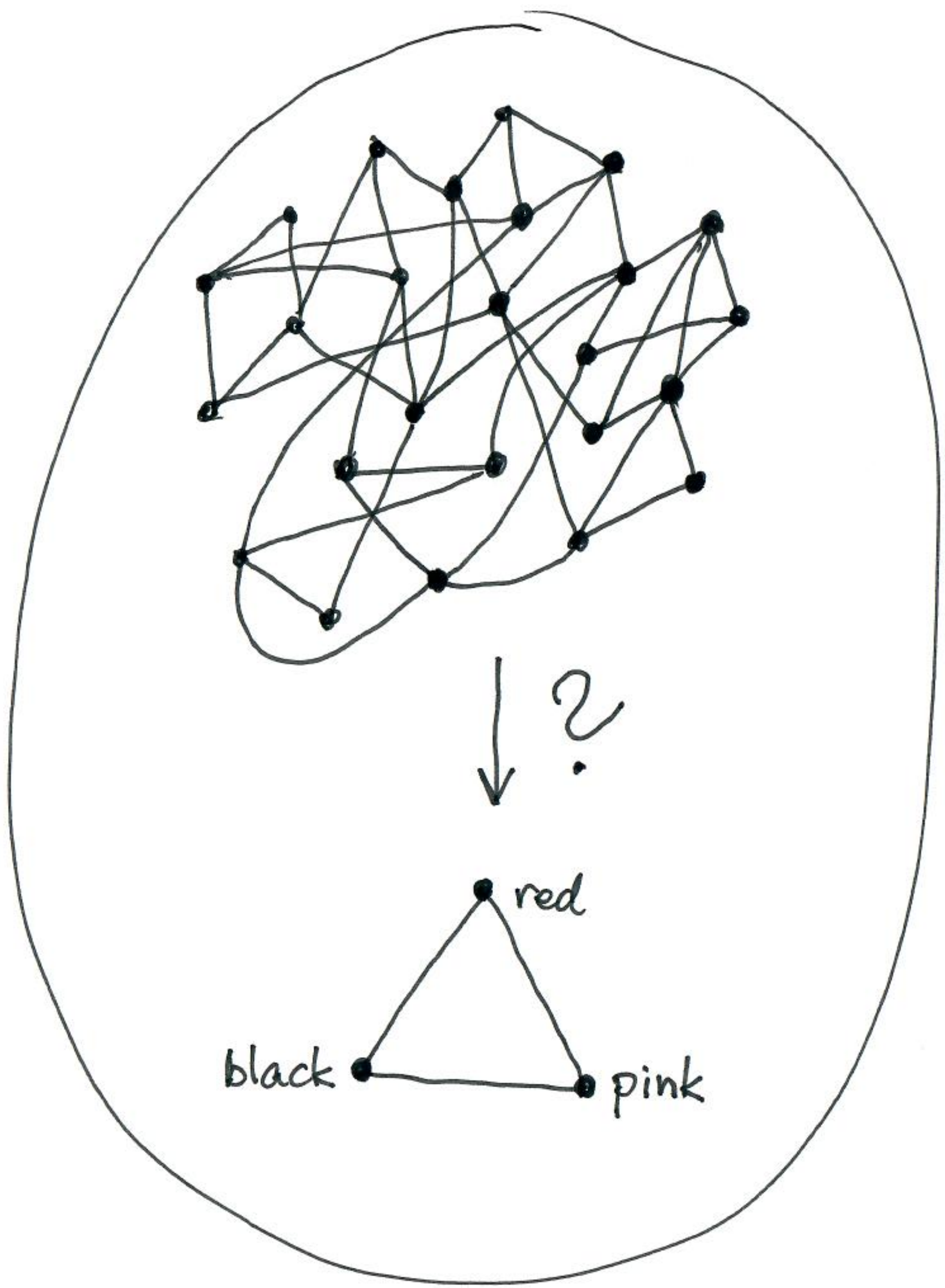
Constraint Satisfaction Problems

# 3-coloring problem

INPUT: graph $\mathbb{X}$

OUTPUT: $\mathbb{X} \longrightarrow \triangle$
(if it exists)

**Question:** how fast can it be solved ?



# A course in computational complexity

<u>computational problem</u>:
- specified class of inputs
- specified correct outputs

<u>examples</u>: the 3-coloring problem, 2-coloring, ....

it is <u>in **P**</u> : can be solved by an algorithm running in time $O(n^{\text{const}})$ where $n$ is the size of the input

<u>in NP</u> : correct answers can be verified in P

<u>NP-complete</u> : hardest in NP

$\boxed{P = NP?}$

# Examples

- 5-coloring
- 3-SAT : Find a satisfying assignment to

  e.g. $(x \vee \neg y \vee \neg u) \wedge (\neg x \vee z \vee w) \wedge (z \vee \check{r} \vee b) \wedge \dots$

- LIN-$\mathbb{Z}_2$ : Find a solution to

  e.g. $\begin{cases} x + y = 1 \\ y + u + w = 0 \\ u + x = 1 \\ \vdots \end{cases}$ in $\mathbb{Z}_2$

- LP : Find a solution to

  e.g. $\begin{cases} 2x + 3y \geq 1 \\ x - 3u + v \leq 5 \\ \vdots \end{cases}$ in $\mathbb{Q}$

# CSP

$\mathbb{A}$ : fixed digraph (or other structure)

- the fixed-template CSP
- many variants

**CSP($\mathbb{A}$)**

INPUT: finite digraph $\mathbb{X}$

OUTPUT: Find $\mathbb{X} \to \mathbb{A}$

- each $\mathbb{A}$ $\rightsquigarrow$ computational problem
- how broad is this class ?
  - general $\mathbb{A}$ : all computational problems
  - finite $\mathbb{A}$ : 3-coloring (  ), 3-SAT, LIN-$\mathbb{Z}_2$

    always in NP

# CSPs

# &

# Symmetries

# Polymorphisms

$$\mathbb{A} = (\underbrace{V}_{\text{vertices}},\ \underbrace{E \subseteq V^2}_{\text{edges}})$$

$f : V^n \longrightarrow V$ is a **polymorphism of $\mathbb{A}$** if

$$\begin{array}{ccccccc} f( v_1, & v_2, & \ldots, & v_n ) & = & w \\ \downarrow & \downarrow & \cdots & \downarrow & & \Rightarrow \downarrow \\ f( v_1', & v_2', & \ldots, & v_n' ) & = & w' \end{array}$$

## Examples

- $$\mathbb{A} = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \qquad f(x, y) = \begin{cases} x & \text{if } x \rightarrow y \\ y & \text{if } x \leftarrow y \end{cases}$$

- $\mathbb{A} = (\mathbb{R}, E \subseteq \mathbb{R}^2 \text{ convex})$ $\qquad f(x, y) = 0.3x + 0.7y$

# CSP and symmetry

**Theorem** [Jeavons '98]

$\mathrm{Pol}(\mathbb{A})$ contains $\mathrm{Pol}(\mathbb{B})$ $\implies$ $\mathrm{CSP}(\mathbb{A}) \leq \mathrm{CSP}(\mathbb{B})$

all polymorphisms of $\mathbb{A}$

no harder than

"the more symmetric the easier"

"complexity depends only on symmetries"

- Improvements: [Bulatov, Jeavons, Krokhin '05]
  [Barto, Opršal, Pinsker '18]
  The Wonderland of Reflections
- Goal: symmetries beyond CSPs

# Endomorphisms vs. polymorphisms

| | endo/auto morphisms | polymorphisms |
|---|---|---|
| what is it | $\mathbb{A} \longrightarrow \mathbb{A}$<br>symmetry of $\mathbb{A}$ | $\mathbb{A}^n \longrightarrow \mathbb{A}$<br>multivariate symmetry of $\mathbb{A}$ |
| trivial | $v \longmapsto v$<br>identity | $(v_1, v_2, \dots, v_n) \longmapsto v_i$<br>$\boxed{\text{dictators}}$ |
| all | endomorphism monoid<br>permutation group | clone |
| studied in | semigroup theory<br>group theory | universal algebra |

# Functional equations

**Theorem** [Bulín, Krokhin, Opršal '19]

CSP($\mathbb{A}$) is equivalent to:

INPUT: trivial system of functional equations* $f(\text{vars}) = g(\text{vars})$

eg.
$$m(x,y,z,x) = f(y,z,x)$$
$$f(x,x,y) = g(y,x)$$
$$m(x,y,x,y) = g(x,y)$$
$$\vdots$$

OUTPUT: solution in Pol($\mathbb{A}$)

trivial = solvable by dictators

* of some fixed large enough bound on arity

# CSP and Symmetry II

Theorem: $\mathrm{CSP}(\mathbb{A}) \sim$ solving trivial systems of special functional equations in $\mathrm{Pol}(\mathbb{A})$

"the more special equations $\mathrm{Pol}(\mathbb{A})$ satisfies, the easier $\mathrm{CSP}(\mathbb{A})$ is"

- only trivial equations $\Rightarrow$ NP-complete
- strong enough equations $\Rightarrow$ in P

"complexity depends only on equations satisfied by symmetries"

clone $\xrightarrow{\text{abstraction}}$ equations among its members

analogues to permutation group $\rightsquigarrow$ group

# CSP history

ancient history

- 2-element structures [Schaefer '78]
- graphs [Hell, Nešetřil '90]
- dichotomy conjecture [Feder, Vardi '98]
  P / NP-complete ?

medieval history

- symmetries [Bulatov, Jeavons, Krokhin, ....]
- describing all homomorphisms [Idziak, Marković, McKenzie, Valeriote, Willard '07]
- consistency [Barto, Kozik '14]
- dichotomy theorem [Bulatov '17, Zhuk '17]

modern history

$\boxed{\text{some nontrivial equations} \Rightarrow \text{in P!}}$

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# Are these shapes symmetric?

1

NO

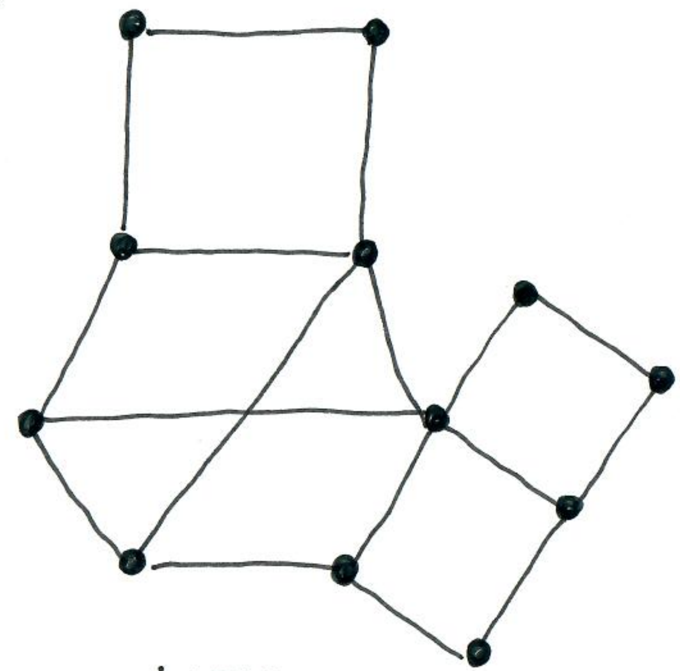
only trivial equations

2

YES

$f(x,y) = f(y,x)$

3



YES

$m(x,x,y) = m(x,x,x)$

$m(x,y,z) = m(y,x,z) =$

$= m(z,y,x) = \dots$

# Analysis
# of
# Symmetries

# Cyclic polymorphism

**Theorem** [Barto, Kozik '12]

*some* nontrivial system of functional equations satisfied in $\mathrm{Pol}(\mathbb{A})$

$\Rightarrow$ *this* "system" is: $f(x_1, x_2, \ldots, x_p) = f(x_2, \ldots, x_p, x_1)$ ($\forall$prime $p > |A|$)

Tool: absorbing subset $B \subseteq A$

$$f(B, B, \ldots, B, A) \subseteq B$$
$$f(B, B, \ldots B, A, B) \subseteq B$$
$$\vdots$$
$$f(A, B, B, \ldots, B) \subseteq B$$

compare

ideal $I \subseteq R$ in ring

$$I \cdot R \subseteq \cancel{R}\, I$$
$$R \cdot I \subseteq I$$

# 3-SAT is hard to approximate

**Theorem** [Håstad '01]

INPUT: e.g. $(x \vee \neg y \vee z) \wedge \overbrace{(\neg x \vee u \vee \neg v)}^{\text{clause}} \wedge (\neg w \vee \neg z \vee \neg r) \wedge \dots$
which is satisfiable

OUTPUT: assignment satisfying $7/8 + \varepsilon$ fraction of clauses

is NP-complete

Tool: Fourier analysis of $\{0,1\}^n \longrightarrow \{0,1\}$

express them in the basis

$x_1, x_2, \dots, x_n$

$x_1+x_2, x_1+x_3, \dots, x_{n-1}+x_n$

$x_1+x_2+x_3, \dots$

$\vdots$

$x_1+x_2+\dots+x_n$

# Promises not helpful for 3-coloring

**Theorem** [Krokhin, Opršal '19]

INPUT: graph $G$ such that $G \to$ 

OUTPUT: find $G \to$ 

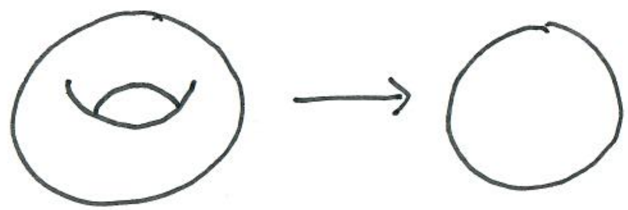
is NP-complete

Tool: algebraic topology

instead of 

topological combinatorics [Lovász '78]

consider 

e.g. for $n=2$ 

# Conclusion

 is **not** symmetric

complexity determined
by symmetry

analysis of symmetries is fun