Optimal control

In this chapter we will be dealing with the problem

$$x' = f(x, u), \tag{1}$$
$$x(0) = x_0, \tag{2}$$

where $x : [0, t] \to \mathbb{R}^n$ is an unknown function, whereas $u(\cdot)$ is a control, which we choose in other to optimize the behaviour of a system in a predetermined sense.

The class of "admissible regulations" typically has the form of

$$\mathcal{U} = \big\{u : (0, t) \to \mathbb{R}^m;\ u \text{ is measurable and } u(s) \in U \text{ for a.e. } s\big\}, \tag{3}$$

where $U \subset \mathbb{R}^m$ is a convex set. It usually holds that $m < n$, i.e. the value of degrees of freedom, that acts on the system, is smaller than the dimension of the whole system.

Let us assume that the properties of the function f can guarantee, that for all $u \in \mathcal{U}$ there exists a unique solution to (1–2) on the interval $[0, t]$. If $x(t) = x_1$ holds for this solution, we will say, that the control u brings x_0 to x_1 in time t, we denote this by

$$x_0 \xrightarrow[u(\cdot)]{t} x_1. \tag{4}$$

In the regulation theory we will most commonly meet the following three types of problems:

1. For a given x_1 and $t > 0$, characterize the set of points x_1 such that $x_0 \xrightarrow[u(\cdot)]{t} x_1$ for some admissible regulation. (Controllability)
2. For a given x_0 a x_1 find an admissible control u such that $x_0 \xrightarrow[u(\cdot)]{t} x_1$, with the minimal time t possible. (Time optimal control)
3. Find $u(\cdot) \in \mathcal{U}$ such that the value of the functional

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(t), u(t))\, dt$$

is maximal. The value $x(T)$ is fixed (more generally it is an element of a given set), whereas time T is arbitrary. Alternatively, we may consider a problem, where time T is fixed, whereas the value $x(T)$ is arbitrary.

Controllability - linear problems

Simple control problems can be solved by elementary considerations. To be able to express ourselves more simply, let us introduce the following notations and notions.

Definition. For $t > 0$ and $x_0 \in \mathbb{R}$ set

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n;\ x_0 \xrightarrow[u(\cdot)]{t} 0 \text{ for a suitable } u(\cdot) \in \mathcal{U} \right\}.$$

where $\mathcal{R}(t)$ is a set of initial conditions, that can be brought by admissible controls to the origin in time t. We call this set the domain of controllability in time t.

The system is called locally controllable in time t, if $\mathcal{R}(t)$ contains a neighbourhood of zero.

Example 1. Show that the system

$$\begin{aligned} x' &= y^3, \\ y' &= u, \qquad u \in [-1, 1], \end{aligned}$$

is locally controllable in the neighbourhood of the origin.

Solution. It suffies to consider, how the solutions behave for values $u \equiv \pm 1$ - they move along the curves

$$\frac{y^4}{4} = \pm x + c.$$

Those solutions fill the whole plane and it is easy to see, that for any $t > 0$ the set $\mathcal{R}(t)$ contains a neighbourhood of zero.

Example 2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function of class C^1 on a neighbourhood of the origin. Then the system

$$x' = f(x)u, \qquad u \in [-1, 1] \tag{5}$$

is not locally controllable for any time $t > 0$.

Solution. Intuitively, the scalar control u only alters the velocity of movement along the curve given by the equation

$$x' = f(x), \qquad x(0) = 0. \tag{6}$$

More accurately: let $X(t)$ be a solution to (6). Then $x(t) := X(\int_T^t u(s)ds)$ is the solution to the former equation (5). Due to the uniqueness this is the only solution (satisfying $x(T) = 0$). Therefore $\mathcal{R}(T)$ contains only points on the trajectory $X(t)$.

Let us now consider the linear case, i.e.

$$x' = Ax + Bu, \tag{7}$$
$$x(0) = x_0, \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices. We choose

$$\mathcal{U} = L^\infty(0,t;\mathbb{R}^m).$$

as the class of admissible controls. The key object of the linear theory is the Kalman matrix of control

$$\mathcal{K}(A,B) = (B, AB, A^2B, \dots A^{n-1}B),$$

which is a $n \times mn$ matrix. The main result of the linear theory is the following theorem.

Theorem 1. *For all $t > 0$, $\mathcal{R}(t)$ is a vector space generated by the columns of the matrix $\mathcal{K}(A,B)$.*

Corollary. *The problem (7) is globaly controllable – i.e. $\mathcal{R}(t) = \mathbb{R}^n$ – for all $t > 0$, if and only if the Kalman matrix $\mathcal{K}(A,B)$ has rank n.*

Remark. Notice that the set $\mathcal{R}(t)$ does not depend on t. That is related to the fact that the values of admissible controls can be arbitrarily large. Therefore it apparently makes no sense to address time optimal controls.

Example 3. Let us consider the system

$$mx'' = u, \tag{9}$$
$$x(0) = x_0,\ x'(0) = y_0.$$

The goal is to choose $u(\cdot) \in L^\infty(0,t)$ such that $x(t) = x'(t) = 0$. The equation describes the (one dimensional) problem of "parking", where m is the mass of the car, the control u is the engine thrust and x_0, y_0 indicate the initial distance from the origin and the velocity.

Let us transform the equation to a first order system for x and $y = x'$:

$$x' = y,$$
$$y' = \frac{u}{m}.$$

In the sense of the general formulation of the system we get $n = 2$, $m = 1$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}.$$

Therefore

$$\mathcal{K}(A,B) = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{pmatrix}$$

and we see that the system is globaly controllable, moreover in an arbitrarily small amount of time (although supposing the unrealistic condition of the arbitrarily large thrust of the engine)

It is unsurprising, that one of the corollaries of the linear theorem is the following local result for unlinear problems.

Theorem 2. *Let $f : V \times U \to \mathbb{R}^n$ be a function of class C^1, let V, U be neighborhoods of the origin in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, let the class of admissible regulations be given by* (3)*. Let (the key assumption) the matrix $\mathcal{K}(A,B)$ have rank n, where*

$$A = \nabla_x f(0,0), \qquad B = \nabla_u f(0,0).$$

Then the equation (1) *is locally controllable for all $t > 0$.*

Remark. The key assumption on the rank of $\mathcal{K}(A,B)$ is not *necessary* as shown in Example 1 above.

Example 4. Let us consider the motion of a pendulum with friction, described by the equation

$$\begin{aligned} mx'' + q(x') + \sin x &= u, \\ x(0) = x_0,\ x'(0) &= y_0. \end{aligned}$$

The function $q(\cdot)$ expresses friction therefore we usually place reasonable physical demands on it. For the purposes of the exercise it will suffice to demand that q is a function of class C^1 and $q(0) = 0$.

Let us transform the equation into a system for x and y, i.e.

$$\begin{aligned} x' &= y, \\ y' &= -\frac{1}{m}\sin x - \frac{1}{m}q(y) + \frac{1}{m}u. \end{aligned}$$

It can be easily solved, the corresponding linearizations are given by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{a}{m} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{m}, \end{pmatrix}$$

where $a = q'(0)$. Then

$$\mathcal{K}(A,B) = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{a}{m^2} \end{pmatrix},$$

which is clearly a regular matrix. The system is therefore locally controllable.

Solve exercices on controllability.

1. Prove that the following points $\tilde{x}$ do not belong to the range of controllability.

(a) $\tilde{x} \in \{(x,y) \in \mathbb{R}^2;\ y > 0\}$

$$\begin{aligned} x' &= u \\ y' &= \cosh x \end{aligned}$$

(b) $\tilde{x} \in R^2 \setminus (0,0)$

$$\begin{aligned} x' &= xy^2u \\ y' &= x^2yu \end{aligned}$$

(c) $\tilde{x} \in \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 \geq 1\}$

$$\begin{aligned} x' &= \frac{x^2}{(x^2+y^2)^{\frac{3}{4}}} - \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} - uy^2 \\ y' &= xy\left(\frac{1}{(x^2+y^2)^{\frac{3}{4}}} - \frac{1}{(x^2+y^2)^{\frac{1}{2}}} + u\right) \end{aligned}$$

for $(x,y) \neq (0,0)$; otherwise $x' = y' = 0$.

(d) $\tilde{x} \in \{(x,y) \in \mathbb{R}^2;\ x^2 + y > 1\}$

$$\begin{aligned} x' &= \begin{cases} \frac{u}{x^2+y-1}, & x^2+y \neq 1 \\ 0, & x^2+y = 1 \end{cases} \\ y' &= x^2 + u^2 \end{aligned}$$

2. Find the set of controllability of the following systems:

(a)

$$\begin{aligned} x' &= xy \\ y' &= \begin{cases} \frac{u^2}{x+y-1}, & x+y \neq 1 \\ 0, & x+y = 1 \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} x' &= \cos(xy) \\ y' &= \cos x + u \end{aligned}$$

3. Determine the area of controllability of the following system

$$\begin{aligned} x' &= xyu \\ y' &= \operatorname{arctg} x - \operatorname{arccotg} y. \end{aligned}$$

How, if at all, will this set change without the demand on the essential boundness of the function u?

4. Without finding exact solutions, design a control procedure of a (globally controllable) system.

$$\begin{aligned} x' &= \sin y \\ y' &= x + u. \end{aligned}$$

5.

$$\begin{aligned} x' &= -x + z \\ y' &= y - z + u \\ z' &= -y + z - u \end{aligned}$$

6. For which choices of the vector $(a, b) \in R^2$ are the following systems globally controllable?

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \mathrm{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} u$$

Consider the following three choices of the matrix A:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

First try to guess the result (and explain it intuitively) based on the behaviour of the system without a control (i.e. set $u = 0$).

7. Show that the equation

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = u$$

is globally controllable.

8. For $n \in \mathbb{N}$ determine the area of control of the system.

$$x' = Ax + Bu,$$

where

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}_{n \times n}$$

and the matrix of control B is as follows

(a)

$$B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n\times 1}$$

(b)

$$B = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}_{n\times 1}$$

9. Let $n \in \mathbb{N}$. Depending on the parameters $\alpha, \beta \in \mathbb{R}$ determine the area of controllability of the system

$$x' = Ax + Bu,$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}_{n\times n}$$

and the control matrix B is in the form

(a)

$$B = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}_{n\times 1}$$

(b)

$$B = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \beta \\ \beta \end{pmatrix}_{n\times 1}$$

10. For $n \in \mathbb{N}$ determine the domain of controllability of the system

$$x' = Ax + Bu,$$

where

$$A = \begin{pmatrix} 0 & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$$

11. For $n \in \mathbb{N}$ determine the domain of controllability of the system

$$x' = Ax + Bu,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n-1 \\ n & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n \times 1}$$

12. Show that the following system is locally controllable in the origin:

$$\begin{aligned} x' &= x + y^2 + u \\ y' &= \sin z + u^2 \\ z' &= x + \sin y + \cos z - 1 \end{aligned}$$

13. Generalize the theorem about local controllability so that the result is controllability of an equation on a neigbourhood of a point $\tilde{x}$ given in advance. Apply this theorem subsequently to the following systems to prove their controllability on a neigbourhood of corresponding points $\tilde{x}$. In some cases it will be necessary to determine correct values of the parameters $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) $\tilde{x} = (\pi/2, 0, \pi)$

$$\begin{aligned} x' &= \sin(\alpha y z) + u^2 \\ y' &= \cos x + \beta u \\ z' &= \operatorname{cotg} x + \cos y + \sin z + \gamma \end{aligned}$$

(b) $\tilde{x} = (1, 1)$

$$\begin{aligned} x' &= -\beta x y + y^\alpha + \beta e^{\frac{\alpha u}{\beta}} - \beta^2 + (\alpha - 1)(\alpha + 1) \\ y' &= \alpha x - 3 + \beta u \end{aligned}$$

(c) $\tilde{x} \in \{(x, y) \in \mathbb{R}^2;\ x \in (-\pi/2, \pi/2) \wedge x - y = \frac{\pi}{2}\}$

$$\begin{aligned} x' &= \alpha \sin x - \beta \cos y - u^2 \\ y' &= \sin^2 x + \beta \cos^2 y + u \end{aligned}$$

(d) $\tilde{x} \in \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 = 4 \wedge xy \geq 0\}$

$$\begin{aligned} x' &= \sqrt{x^2 + y^2} - x^2 - y^2 + ux \\ y' &= e^{x^2 + y^2} + uy \end{aligned}$$

Hints and solutions.

1) (a) $y' \geq 1$.

(b) Let us notice that $(x^2 - y^2)' = 0$, therefore we can immediately discard points (x, y) that do not satisfy the equation $x^2 = y^2$ (see Figure 1). Non-controllability (1-dimensional) of the equation $x' = x^3 u$ can be shown by integration and using the essential boundness of u.

(c) Following the transition into polar coordinates, to which we are encouraged by radial elements, we get

$$\begin{aligned} r' &= \sqrt{r}(1 - \sqrt{r}) \cos \omega \\ \omega' &= ru \sin \omega. \end{aligned}$$

If the solution penetrates the unit circle, it will not leave it.
If we do not want to use polar coordinates, it is possible to multiply the first equation by x, the second by y, sum up both equations and notice that on the unit circle $(x^2 + y^2)' = 0$ holds.

(d) If the solution gets to the dividing parabola, it can only contiune in the positive direction of the y axis.

2) (a) First we plot the given vector field (see Figure 2). After a short consideration we discard everything apart from the set $\{(0, s);\ s \in (0, 1)\}$, after that it is sufficient to choose $u \equiv 1$.

(b) Following the graphical depiction (it is important to plot the curve $xy = \frac{\pi}{2} + k\pi$ correctly, see Figure 3) we deduce that the set $\{(x, y) \in \mathbb{R}^2;\ x < 0\}$ is the area of controllability.
An example of a control procedure is to first set $u \equiv -2 \operatorname{sgn} y_0 + y_0/x_0$. This way we will meet the trajectory of the controled solution, which exists in the set $\{(x, y) \in \mathbb{R}^2;\ |xy| \leq \pi/4 \wedge x < 0\}$ and going backwards in time to infinity it has the limit $(0, -\infty)$.

3) On the set $\{(x, y) \in \mathbb{R}^2;\ y = 0\}$ we observe $y' < 0$ (see Figure 4) and on a certain neigbourhood of the origin we have $y' \leq c < 0$. From this we get the non-controllablity of $\{(x, y) \in \mathbb{R}^2;\ y \leq 0\}$. On the other hand the set $\{(0, y);\ y > 0\}$ is controllable inependently of the choice of u. On the remaining areas it holds that $|x'| \leq |x| \cdot y_0 \cdot \|u\|_\infty$ due to y being decreasing close to the x-axis. The corresponding solutions are therefore pushed off of zero by the function $x_0 \exp(-t \cdot y_0 \cdot \|u\|_\infty)$ in the x-coordinate.
If we abandon the demand $\|u\|_\infty < \infty$, we are still unable to improve the situation of $\{(x, y) \in \mathbb{R}^2;\ y \leq 0\}$ for the same reasons as in the previous case. However, the same does not apply to $\{(x, y) \in \mathbb{R}^2;\ x \neq 0,\ y > 0\}$. By choosing u as a piecewise constant function ("stairs to the sky"), we are able

to maintain $|x'| \geq c > 0$, where c is a constant big enough so that the solution approaches the y-axis faster than the x-axis. (e.g. $c = 3\pi|x_0|y_0^{-1}/2$). Due to the limit of a monotone sequence such a solution must necessarily end up on the y-axis in time bounded by the constant c. By weakening the assumptions we have extended the domain of controllability to $\{(x, y) \in \mathbb{R}^2;\ y > 0\}$.

4) We will show the idea in three steps. By their combination, we will control the solution for any initial condition. To better visualize this see Figures 5 and 6.

I/$(x_0, y_0) = (0, 2k\pi),\ k \in \mathbb{N}$.

Set $u \equiv -1$. Since

$$\left(\frac{(x-1)^2}{2} + \cos y\right)' = 0$$

holds, the solution with the initial condition satisfies $x = 1 - \sqrt{3 - 2\cos y}$, which is a 2π-periodical function in y, strictly negative on $(2(k-1)\pi, 2k\pi)$. From this and the equation for y' we get that the solution in question will reach $(0, 2(k-1)\pi)$ no later than at $t = 2\pi$. See also that

$$C := \max_{y \in [2(k-1)\pi, 2k\pi]} |x(y)| = \sqrt{5} - 1.$$

Analogous process with $u \equiv 1$ will ensure the passing $(0, -2k\pi) \to (0, -2(k-1)\pi)$.

II/$(x_0, y_0) \in \{(x, y) \in \mathbb{R}^2;\ y \in [2k\pi + 5\pi/4, 2k\pi + 7\pi/4],\ k \in \mathbb{Z} \wedge (x, y)$ lies on the right of the trajectory of the previous point in between $(0, 2(k+1)\pi)$ and $(0, 2k\pi)\}$.

The interval $[2k\pi+5\pi/4, 2k\pi+7\pi/4]$ marks a strip in which $x' \leq -1/\sqrt{2}$. By a suitably switching u we will stay in this interval with the y-coordinate. We have achieved a drift in the x-coordinate up to the touch with the trajectory from the previouse point, which is when we will involve the corresponding value of u. The time continuation of this phase won't exceed the value $\sqrt{2}(|x_0| + C)$.
An analogous method will work for (x_0, y_0) such that $y_0 \in [2k\pi + \pi/4, 2k\pi + 3\pi/4],\ k \in \mathbb{Z}$, and moreover (x_0, y_0) lies on the left of the trajectory of the previous point in between $(0, 2(k+1)\pi)$ and $(0, 2k\pi)$.

III/(x_0, y_0) everywhere else.

By plugging a suitable constant for u we will reach "the pipe" from the previous case or we will cross the trajectory from the first case. E.g. by choosing $u \equiv -x_0 + 2\pi$ it will happen in time $t < 3$.

5) The Kalman matrix has rank 2 and its columns generate the hyperplane $y + z = 0$.

6) (i) $a^2 + b^2 \neq 0$, (ii) $ab \neq 0$, (iii) $b^2 + 2ab - a^2 \neq 0$.

7) Transform the equation to a system of n equations; the Kalman matrix has ones on the adjacent diagonal and is null above it.

8) (a) $\mathcal{K}(A, B)$ is a lower triangular matrix with ones on the diagonal.

(b)

$$\mathcal{K}(A, B) = \begin{pmatrix} 1 & 2 & \dots & 2^{n-1} \\ 1 & 2 & \dots & 2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & 2^{n-1} \end{pmatrix}$$

The columns of the Kalman matrix generate $\mathrm{lin}\{(1, 1, \dots, 1)\}$.

9) (a)

$$\mathcal{K}(A, B) = \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 & 0 \\ 0 & \alpha & \beta & \dots & 0 & 0 \\ 0 & 0 & \alpha & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha & \beta \\ \beta & 0 & 0 & \dots & 0 & \alpha \end{pmatrix}_{n \times n}$$

If n is odd then: $\alpha \neq -\beta \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n$.
$\alpha = -\beta \neq 0 \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n - 1$, the columns of $\mathcal{K}(A, B)$ generate the hyperplane $(1, \dots, 1)^{\perp}$.

If n is even then: $\alpha \neq \pm\beta \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n$.
$\alpha = -\beta \neq 0 \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n - 1$, the columns of $\mathcal{K}(A, B)$ generate the hyperplane $(1, \dots, 1)^{\perp}$.
$\alpha = \beta \neq 0 \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n - 1$, the columns of $\mathcal{K}(A, B)$ generate the hyperplane $(1, -1, \dots, 1, -1)^{\perp}$.

(b)

$$\mathcal{K}(A, B) = \begin{pmatrix} \alpha & \beta & \beta & \dots & \beta \\ \beta & \alpha & \beta & \dots & \beta \\ \beta & \beta & \alpha & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots \\ \beta & \beta & \beta & \dots & \alpha \end{pmatrix}_{n \times n}$$

$\alpha = \beta \neq 0 \Rightarrow$ the columns of $\mathcal{K}(A, B)$ generate $\mathrm{lin}\{(\alpha, \alpha, \dots, \alpha)\}$.
$\alpha = -(n - 1)\beta \neq 0 \Rightarrow \mathrm{h}(\mathcal{K}(A, B)) = n - 1$, the columns generate the hyperplane $(1, \dots, 1)^{\perp}$.

10)

$$
\mathcal{K}(A,B) = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}_{n\times n}
$$

The Kalman matrix is regular, which we can check by calculating its determinant. After drawing a recurent formula and checking a guess derived from $n = 1, 2, 3$ we get $\det(\mathcal{K}(A,B)) = n + 1$.

11)

$$
\mathcal{K}(A,B) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & (n-1)! \\ 0 & 0 & 0 & 0 & \dots & (n-1)! & n! \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & & 0 & (n-3)(n-2)(n-1) & \dots & 0 & 0 \\ 0 & 0 & (n-2)(n-1) & (n-2)(n-1)n & \dots & 0 & 0 \\ 0 & n-1 & (n-1)n & 0 & \dots & 0 & 0 \\ 1 & n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}
$$

The Kalman Matrix is regular (again, we can easily calculate the determinant $\det(\mathcal{K}(A,B))$).

13) The only modification is in the assumption $f(\tilde{x}, 0) = 0$ and in using $\nabla_x f(\tilde{x}, 0), \nabla_u f(\tilde{x}, 0)$, where $x' = f(x, u)$ – generalization of $f(0, 0) = 0$, $\nabla_x f(0, 0)$, $\nabla_u f(0, 0)$. In the following we denote $A = \nabla_x f(\tilde{x}, 0), B = \nabla_u f(\tilde{x}, 0)$.

(a)

$$
\mathcal{K}(A,B) = \begin{pmatrix} 0 & \alpha\beta\pi & 0 \\ \beta & 0 & -\alpha\beta\pi \\ 0 & 0 & \alpha\beta\pi \end{pmatrix}
$$

$\alpha \neq 0, \beta \neq 0, \gamma = -1.$

(b)

$$
\mathcal{K}(A,B) = \begin{pmatrix} \alpha & -\beta^2 \\ \beta & \alpha^2 \end{pmatrix}
$$

$\alpha = \beta = 3.$

(c)

$$
\mathcal{K}(A,B) = \begin{pmatrix} 0 & \cos x \\ 1 & \sin(2x) \end{pmatrix}
$$

$\alpha = \beta = -1.$

(d)

$$
\mathcal{K}(A,B) = \begin{pmatrix} x & -6 \\ y & 8e^4 \end{pmatrix}
$$

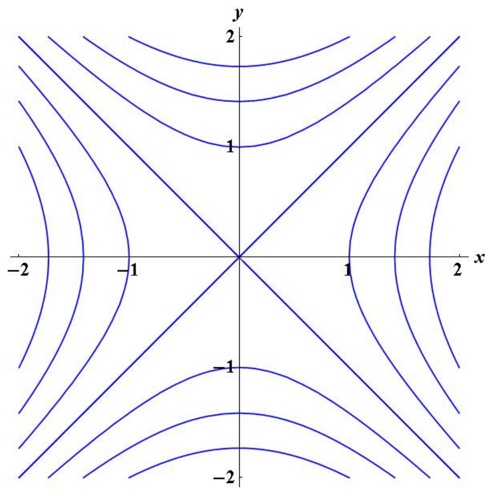

Figure 1: Exercise 1b – level sets of the function $x^2 - y^2$ for values $-3, -2, \dots, 3$.

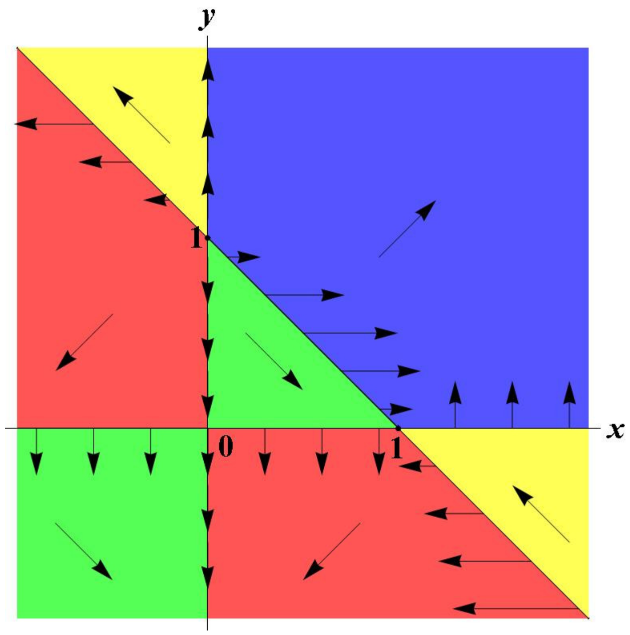

Figure 2: Phase portrait of Exercise 2a, where equal colours denote the equal combination of signs of x' and y'.

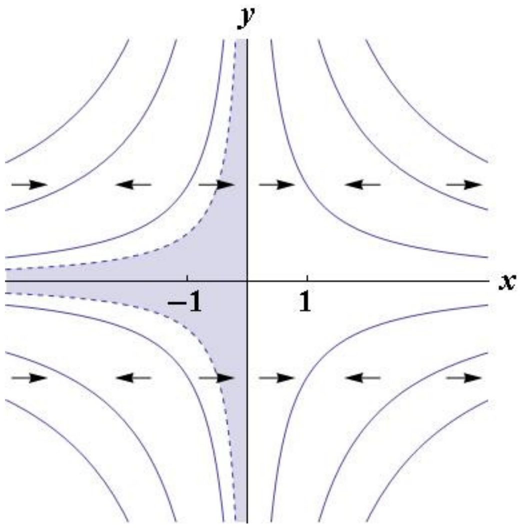

Figure 3: Exercise 2b – level sets of the function xy for values $-5\pi/2$, $-3\pi/2 \ldots 5\pi/2$ with the x' direction depicted. The highlighted set is $\{(x, y) \in \mathbb{R}^2;\ |xy| \leq \pi/4 \wedge x < 0\}$.

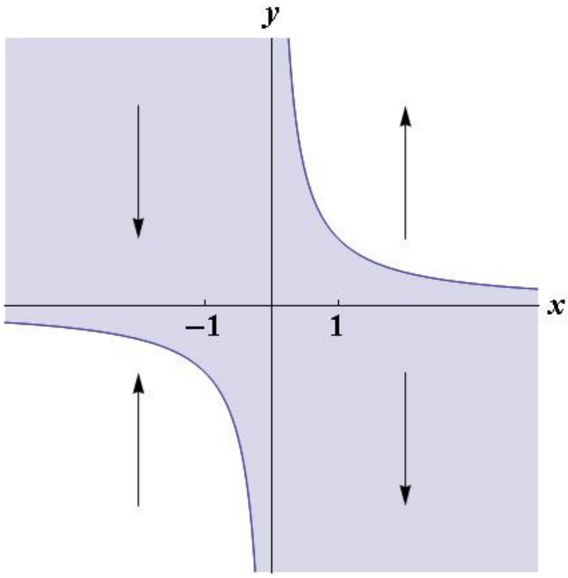

Figure 4: An aid to Exercise 3, where the highlighted depicts $y' < 0$.

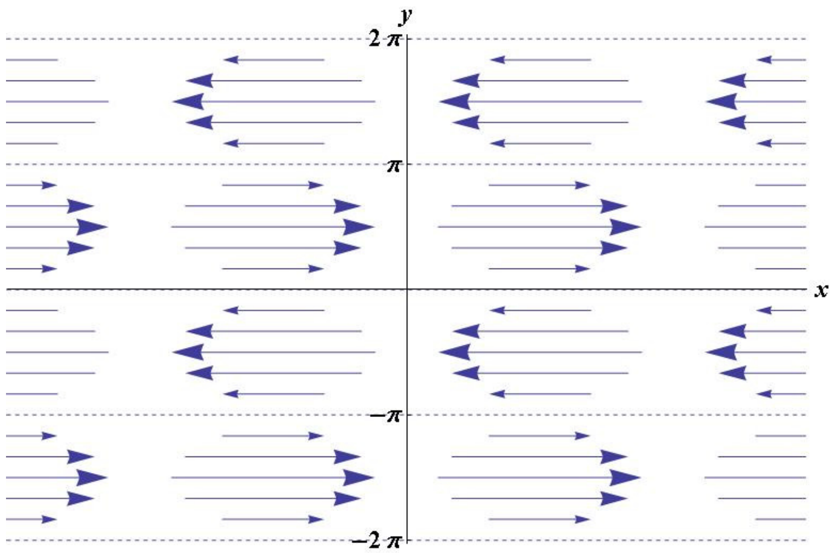

Figure 5: A depiction of x' from Exercise 4.

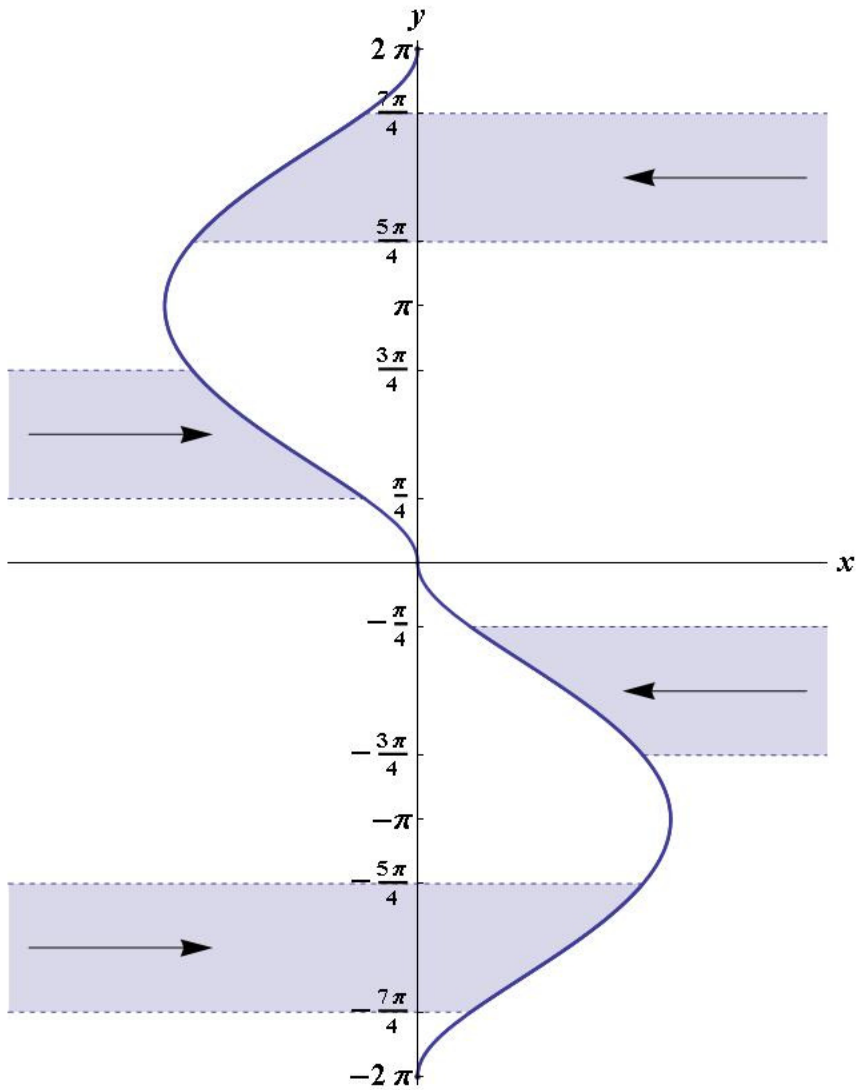

Figure 6: The trajectory of controled solutions that begin in $(0, -2\pi)$ and $(0, 2\pi)$ from exercise 4. The highlighted regions depict the initial conditions which fall under step II.

Observability.

Let us now consider a general non-linear equation

$$x' = f(x) \tag{10}$$

and let us define an ,,observed variable”

$$y = g(x), \tag{11}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$. Again, it usually holds $m < n$, i.e. the observation contains less information than the whole system.

Definition. We say, that the equation (10) is observable via the quantity (11), if for any two solutions x_1, x_2 and a time $t > 0$ it holds:

$$g(x_1) = g(x_2) \text{ on } [0, t] \qquad \Longrightarrow \qquad x_1(0) = x_2(0).$$

Remark. Considering the uniqueness of the solution, the conclusion of the implication $x_1(0) = x_2(0)$ is equivalent to the solution being equal on the whole interval $[0, t]$.

Observability exercises can again be solved by elementary considerations. In the linear case we get a general solution; moreover it can be seen that observability is in a certain way a dual term to controllability.

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ be constant matrices. Then the equation*

$$x' = Ax \tag{12}$$

is observable via

$$y = Bx, \tag{13}$$

if and only if the equation

$$x' = A^T x + B^T u$$

is globally controllable.

Corollary. *The equation* (12) *is observable via* (13), *if and only if* $\mathcal{K}(A^T, B^T)$ *has rank n.*

Example 5. Find a necessary and sufficient condition for the numbers a_{ij}, such that the system

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned}$$

is observable in the quantity x.

Solution. In concord with the previous theorem we have

$$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \qquad B^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which means

$$\mathcal{K}(A^T, B^T) = \begin{pmatrix} 1 & a_{11} \\ 0 & a_{12} \end{pmatrix};$$

this matrix has the required rank 2, if and only if $a_{12} \neq 0$.

Solve the following exercises on observability.

14. Find (the simplest) example of a matrix A such that the system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \mathrm{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is observable via the quantity x, eventually characterize such matrices.

15. Depending on $m, n \in \mathbb{N},\ m < n$, determine, for which $V \in \mathbb{R}^{m \times n}$ will the system $x' = Ax$ be observable via the quantity $y = Vx$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 4 & 6 & \dots & 2n \\ 3 & 6 & 9 & \dots & 3n \\ \dots & \dots & \dots & \dots & \dots \\ n & 2n & 3n & \dots & n^2 \end{pmatrix}$$

16. Examine the observability of the following systems via V_1 and V_2:

(a) $x' = y^2$
$y' = x^2$

$V_1 = x - y$
$V_2 = x$

(b) $x' = y^2$
$y' = -x^{-4}$

$V_1 = x \cdot y$
$V_2 = x$

17. Consider the systems

$$x' = xy$$
$$y' = -y/x.$$

Determine whether it is observable via the quantity $V = x \cdot y$, if we only suppose the following initial conditions (x_0, y_0):

(a) $(x_0, y_0) \in \mathbb{R}^2 \setminus \mathrm{lin}\{e_2\}$

(b) $(x_0, y_0) \in \{(a, a) |\ a \in \mathbb{R} \setminus \{0, \pm 1\}\}$

18. Without refering to the controllability theory, determine the observability of the equation $x''' - x'' + x' - x = 0$ via the quantities:

(a) $V = x + x''$

(b) $V = x + x' + x''$

(c) $V = (x + x'')x'$

19. Prove, that the system

$$x' = y$$
$$y' = e^x$$

is not observable via the quantity $V = \sin \frac{1}{y}$.

(hint: focus on the case $y_0 = \sqrt{2e^{x_0}}$)

20. Consider the linear system

$$x' = y$$
$$y' = -x,$$

$$(x, y) = (x(t), y(t)), \quad t \in [0, \pi].$$

Examine its observability via the given quantities. In case of non-observability, characterize all the solutions, which the given quantity can not separate.

(a) $V = x^2 + y^2$
(b) $V = x$
(c) $V = x \cdot y$
(d) $V = x(0) \cdot y(0)$
(e) $V = S \cdot (x - y)$,

where $S = S(t)$ is the area of the region delimited by the curve $\{(x(s), y(s)),\ s \in [0, t]\}$ and the lines connecting its end points with the origin $(0, 0)$.

Solutions

14) First row $(0,1,0)$, second row $(0,0,1)$. It can not have a smaller rank than A.

15) The columns of the matrices $A(= A^T), A^2, \ldots, A^{n-1}$ are multiples of the vector $(1,\ldots,n)$ and therefore the columns of $\mathcal{K}(A^T, V^T)$ generate $\operatorname{lin}\{v_1,\ldots,v_m,(1,2,\ldots,n)\}$, where v_i are the rows of the matrix V. The system is observable via V if and only if $m = n-1$ and $\{v_1,\ldots,v_m,(1,2,\ldots,n)\}$ is a linearly independet set.

16) (a) $x_0 = y_0 \Rightarrow V_1 \equiv 0$, i.e. the system is not observable via V_1. The system is however observable via V_2. We can prove by contradiction ($x^1 \equiv x^2 \wedge y_0^1 \neq y_0^2$), we will use that y is non-decrasing.

(b) $x_0 = 1/y_0 \Rightarrow V_1 \equiv 1$ and therefeore the system is not observable via V_1. Observability via V_2 can be proved as in the previous system.

17) (a) $x_0 = 1/y_0 \Rightarrow V_1 \equiv 1$, the system is not observable.

(b) Set $c = (x_0^2 - 1)/x_0$. Then the solution of the system is

$$x(t) = \frac{x_0^2 e^{ct} - 1}{c}$$
$$y(t) = \frac{c x_0^2 e^{ct}}{x_0^2 e^{ct} - 1}$$

From the quantity $V = x \cdot y = x_0^2 e^{ct}$ we can determine a unique x_0, and therefore the system is observable for the given initial conditions.

18) $x(t) = c_1 e^t + c_2 \sin t + c_3 \cos t$,
where $c_{1,2,3} = c_{1,2,3}(x_0, x_0', x_0'')$ is a one to one function $\mathbb{R}^3 \to \mathbb{R}^3$.

(a) $V = x + x'' = 2c_1 e^t$,
and therefore V does not distinguish the solution from the identical c_1. The equation is not observable via V.

(b) $V = x + x' + x'' = 3c_1 e^t + c_2 \cos t - c_3 \sin t$,
from which we can uniquely determine (c_1, c_2, c_3) (therefore also (x_0, x_0', x_0'')). The equation is observable via V.

(c) V does not distinguish solutions with oposite signs, i.e. the equation is not observable via V.

19) If we consider the case $y_0 = \sqrt{2e^{x_0}}$, the given system is solved by the functions

$$x(t) = \ln\left(\frac{2}{2 - y_0 t}\right)^2 + x_0$$
$$y(t) = \frac{2y_0}{2 - y_0 t}$$

from which

$$V = \sin\frac{1}{y} = \sin\left(\frac{1}{y_0} - \frac{t}{2}\right) = \sin\left(\frac{1}{\sqrt{2e^{x_0}}} - \frac{t}{2}\right).$$

If we select a second initial condition as

$$x_1 = 2\ln\frac{\sqrt{e^{x_0}}}{1 + 2\pi\sqrt{2e^{x_0}}},$$

then the solutions will not be distinguishible via V because of the sine periodicity.

20) The system is solved by

$$\begin{aligned} x(t) &= r_0 \sin(t + \omega_0) \\ y(t) &= r_0 \cos(t + \omega_0), \end{aligned}$$

where $(x(0), y(0)) = (r_0 \sin\omega_0, r_0\cos\omega_0)$, $r_0 \geq 0$, $\omega_0 \in [0, 2\pi)$. Considering the restriction of the domain to $[0, \pi]$ the graph is a semicircle with its center in the origin and it is drawn in a constant speed in the clockwise orientation.

(a) non-observable
Every solution with the identical quantity r_0 coincide.

(b) observable
By controllability theory or by an elementary consideration.

(c) non-observable
The solutions (x, y) and $(-x, -y)$ can not be distinguished; i.e. the quantity $V = \frac{r_0^2}{2}\sin(2t + 2\omega_0)$ determine ω_0 except for a multiple of π, however we would need to determine ω_0 except for a multiple of 2π.

(d) observable
The coinciding solutions are those for which $(x(0), y(0)) = (x(0), V/x(0))$ hold for $V \neq 0$ and with the initial conditions on the axes x, y for $V = 0$.

(e) observable
$S(t) = \frac{r_0^2}{2}t$ and using a suitable sum formula (for trigonometric functions) we get

$$V = -\frac{r_0^3}{\sqrt{2}}t\cos\left(t + \omega_0 + \frac{\pi}{4}\right),$$

from which we can uniquely determine (r_0, ω_0) on $[0, \pi]$.

Time optimal control. Maximum principle.

Let us once again consider the linear problem

$$x' = Ax + Bu, \tag{14}$$

however for bounded values of admissible controls only. More precisely, we want

$$u(\cdot) \in \mathcal{U} = \big\{u : (0,t) \to U \text{ measurable, } U = [-1,1]^m\big\} \tag{15}$$

The goal is to choose $u(\cdot)$ such that $x_0 \xrightarrow[u(\cdot)]{t} 0$ in the shortest amount of time possible.

Let us first entertain questions of controllability of the exercise (14–15). Let us once again set

$$\mathcal{R}(t) = \big\{x_0 \in \mathbb{R}^n;\ x_0 \xrightarrow[u(\cdot)]{t} 0 \text{ for suitable } u(\cdot) \in \mathcal{U}\ \big\}$$

and then let us define

$$\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t).$$

Then the following theorem holds.

Theorem 4. *Let the matrix $\mathcal{K}(A,B)$ have rank n. Then for every $t > 0$ the problem (14–15) is locally controllable, i.e. $\mathcal{R}(t)$ contains a neighbourhood* 0.

If in addition $\operatorname{Re}\lambda \le 0$ *for all eigenvalues λ of the matrix A, the problem is globally controllable, i.e. $\mathcal{R} = \mathbb{R}^n$.*

Deciding on the *existence* of the time optimal control is simple – thanks to the linearity of the equation and the convexity of the set U.

Theorem 5. *Let $x_0 \in \mathcal{R}(t)$ for some $t > 0$. Then there exists a $t^* \le t$ and $u^*(\cdot) \in \mathcal{U}$ such that $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$, where time t^* is the smallest possible.*

The following theorem gives us a *necessary* condition for the optimality of a control.

Theorem 6 (Pontryagin maximum principle)**.** *Let $u(\cdot) \in \mathcal{U}$ bring $x_0 \in \mathcal{R}^n$ to* 0 *in the optimal – i.e. smallest possible – time t. Then there exists a non-zero vector $h \in \mathbb{R}^n$ such that*

$$h^T \exp(-sA)Bu(t) = \max_{\eta\in[-1,1]^m} h^T \exp(-sA)B\eta \tag{16}$$

for almost every $s \in (0,t)$.

Remark. Let us recall the method of Lagrange multipliers. The necessary condition for x to be an extremum point of a function $f(x)$ in the set $\{x;\ g_1(x) = 0,\ \dots,\ g_k(x) = 0\}$, is – considering suitable assumptions on the smoothness of functions f, g_j – the existence of numbers $\lambda_1, \dots, \lambda_k$ such that

$$\nabla g(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_k \nabla g_k(x). \tag{17}$$

When solving exercises, we usually do not have to compute the values of λ_j; we get some information from the equation (17), thanks to which we restrict the set of ,,suspected" points to just a few points. With the information about the *existence* of extrema, we easily identify the ,,culprit".

The situation here is simmilar: the condition (16) seems rather mysterious, however in concrete cases it easily gives enough information for us to identify the form of the optimal control.

Exercise 3 – continuation. Let us find a control $u : (0, t) \to [-1, 1]$, that will park in the smallest amount of time possible. Its existence is ensured by Theorems 4, 5 together with the spectrum of A containing only the value 0. It can be easily computed from the definition that

$$\exp(-sA) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}.$$

By the Pontryagin maximum principle there exists a non-zero vector $h = (h_1, h_2)$ such that

$$(h_2 - h_1 s) u(s) = \max_{\eta \in [-1,1]} (h_2 - h_1 s)\eta.$$

From this we easily get that $u(s) = \operatorname{sgn}(h_2 - h_1 s)$ for almost every s. Specially this means that u is only equal to ± 1 and the change of sign occurs at most once. In other to construct an optimal control, it is useful to sketch out the behaviour of the system for $u = \pm 1$. It can easily be shown that the corresponding first integrals are parabolas

$$\pm x = c - \frac{m}{2}(x')^2.$$

It is best to construct the optimal control in reverse: for $u = -1$ we see the solution in Figure 7, which brings the system in finite time to the origin (in blue). We get to this trajectory by $u = 1$ (in red). It is not difficult to figure out, that any initial condition can be controlled in this *unique* way.

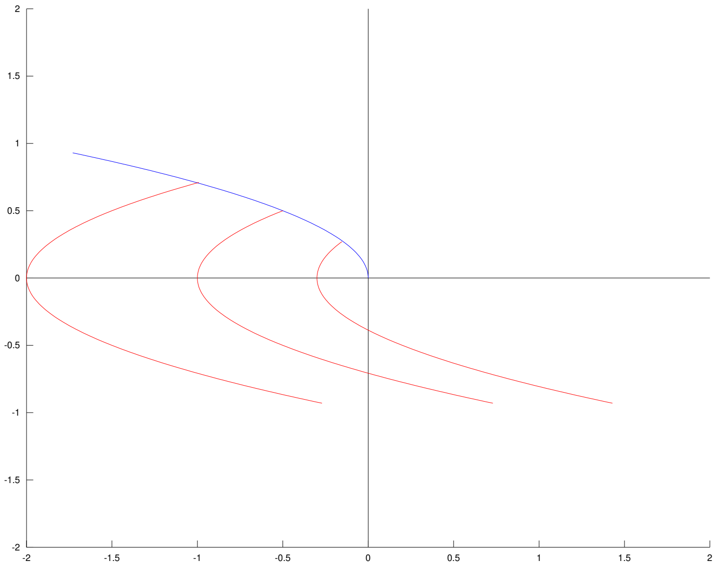

Figure 7: Exercise 3 – solutions for $u = -1$ (in blue) and $u = 1$ (in red).

Maximum principle – general case.

Finally let us formulate the Pontryagin maximum principle in a general form as a *necessary* condition for optimal control.

Let us consider the general problem

$$x' = f(x, u), \tag{18}$$

with admissible controls in the form

$$u(\cdot) \in \mathcal{U} = \big\{u : (0, T) \to U;\ u \text{ is measurable and } u(s) \in U \text{ for a.e. } s\big\}, \tag{19}$$

where $U \subset \mathbb{R}^m$ is an arbitrary set. The goal is to maximize the functional

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(t), u(t))\, dt. \tag{20}$$

Let us introduce the following notions. The Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as

$$H(x, p, a) = f(x, a) \cdot p + r(x, a)$$

and the adjoint problem

$$p' = -\nabla_x H(x, p, u) \tag{21}$$

Theorem 7. *Let $u(\cdot)$ be the maximizer of the problem (18–20), where time $T > 0$ and the initial condition $x(0) = x_0$ are given, whereas $x(T)$ is not. Let us assume, that the functions f, r and g are continuous and have continuous derivatives with respect to x.*

Then for almost every $s \in (0, T)$ the equality

$$H(x(t), p(t), u(t)) = \max_{\eta \in U} H(x(t), p(t), \eta)$$

holds, where $p(t)$ is a solution of the adjoint problem (21) *with the* final *condition*

$$p(T) = \nabla_x g(x(T)). \tag{22}$$

The assumptions of the theorem guarantee that for a given $u(\cdot) \in \mathcal{U}$ there exists a unique solution $x(t)$ of the equation (18). Coordinates of the adjoint problem (21), (22) satisfy

$$p_i' = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(x(t), u(t)) p_j - \frac{\partial r}{\partial x_i}(x(t), u(t)), \qquad p_i(T) = \frac{\partial g}{\partial x_i}(x(T)).$$

That is a linear equation and therefore has – for given $x(t)$, $u(t)$ – a unique solution $p(t)$ on the interval $[0, T]$.

Example 6. The national economy follows the equation

$$x' = kxu,$$

where x is the total capital, $k > 0$ is the constant that expresses the natural rate of growth and

$$u : (0, T) \to [0, 1]$$

expresses the percentage of reinvestments i.e. $1 - u$ is the part of production that is consumed. The goal is to choose u such that *the total* consumption

$$P[u(\cdot)] = \int_0^T (1 - u(t)) x(t)\, dt$$

is maximal. Time $T > 0$ is fixed; the quantity $x(T)$ is arbitrary.

Solution. In the sense of the established setting the Hamiltonian is equal to

$$H = x\big(1 + a(pk - 1)\big).$$

The Pontryagin maximum principle then says that in case of an optimal solution we have

$$x(t)\big(1 + u(t)(p(t)k - 1)\big) = \max_{\eta \in [0,1]} x(t)\big(1 + \eta(p(t)k - 1)\big).$$

It is sensible to assume $x(0) > 0$, from which $x(t) > 0$ for all $t \geq 0$. It is sufficient to maximize the second parenthesis; from this we deduce that the optimal control satisfies

$$u(t) = \begin{cases} 0, & \text{if } p(t)k < 1, \\ 1, & \text{if } p(t)k > 1. \end{cases}$$

Now we need to find a solution to the adjoint problem. Since $\frac{d}{dx}H = 1 + a(pk-1)$ we have the equation

$$p' = \begin{cases} -1, & \text{if } p(t)k < 1, \\ -pk, & \text{if } p(t)k > 1. \end{cases}$$

For our problem $g = 0$ and therefore the corresponding final condition is

$$p(T) = 0.$$

Now (backtracking from $t = T$) we compute that

$$p(t) = \begin{cases} T - t, & t \in [T - \frac{1}{k}, T], \\ \exp\big(k(T-t) - 1\big), & t < T - \frac{1}{k}. \end{cases}$$

In total we get: if $\frac{1}{k} < T$, it is optimal to choose $u = 1$ for $t \in [0, T - \frac{1}{k}]$ and $u = 0$ pro $t \in [T - \frac{1}{k}, T]$. In case that $\frac{1}{k} \geq T$ we always choose $u = 0$.

Example 7. Let us consider the equation $x' = x/u$, $x(0) = 1$. Find the necessary condition on $u : [0, T] \to [1, 3]$, so that $P[u(\cdot)] = \int_0^3 x(t)u(t)dt$ is maximal.

Solution. The hamiltonian is $H = x(p/u + u)$. From the linearity of the equation in x and the initial condition we get that $x(t) > 0$; therefore the maximum condition can be written as follows

$$\frac{p(t)}{u(t)} + u(t) = \max_{a \in [1,3]} \frac{p(t)}{a} + a.$$

Let us examine the course of the function $h(a) = \frac{p_0}{a} + a$, $a > 0$, depending on $p_0 \in \mathbb{R}$. For $p_0 \leq 0$, $h(a)$ is strictly increasing. For $p_0 > 0$, $h(a)$ is strictly convex with a global minimum at $a = \sqrt{p_0}$. In both cases the maximum with respect to $a \in [1, 3]$ is in one of the extremal points of the interval. By plugging in we can easily compute that for $p_0 > 3$ we get $a_{max} = 1$, whereas for $p_0 < 3$ we get $a_{max} = 3$.

From the maximum principle we get, that $u(t) = 1$ if $p(t) > 3$, whereas $u(t) = 3$ for $p(t) < 3$.

The adjoint equation looks as follows

$$p' = -\frac{p}{u(t)} - u(t), \qquad p(3) = 0,$$

since $g = 0$. The solution will once again be constructed ,,backwards". Set

$$t_0 = \inf \big\{ t \in [0,3];\ p < 3 \text{ na } [t,3] \big\}.$$

From continuity we get that $t_0 < 3$ and clearly $p(t) < 3$ and therefore $u(t) = 3$ on $(t_0, 3]$. Therefore we have the equation $p' = -p/3 - 3$, which has a general solution $p = ce^{-t/3} - 9$. From the assumption $p(3) = 0$ we get $c = 9e$. In total we have

$$p = 9e^{1-t/3} - 9, \quad t \in (t_0, 3]. \tag{23}$$

From the definition of t_0 we get that $p(t_0+) = 3$, or $t_0 = 0$. The function (23) is equal to 3 at $t = 3 - 3\ln 4/3 > 0$, therefore necessarily

$$t_0 = 3 - 3\ln 4/3. \tag{24}$$

p is clearly decreasing and positive on the interval $[0, t_0)$; therefore in particular we have here $p > 3$ and $u = 1$. The adjoint equation transforms into $p' = -p - 1$. The general solution is $p = ce^{-t} - 1$, from the condition $p(t_0) = 3$ we get $c = 4(3/4)^3 e^3$. In total

$$p = 4\left(\frac{4}{3}\right)^3 e^{3-t} - 1, \quad t \in [0, t_0). \tag{25}$$

The important equation is however about the optimal control, i.e. $u = 1$ on $(0, t_0)$ and $u = 3$ on $(t_0, 3)$. – Let us stress, that we only showed, that if an optimal control exists, it must have the stated form. The existence of the maximum is non-trivial (because of the non-linearity of the problem). From this we can easily compute the solution: $x = e^t$ for $t \in [0, t_0]$ and $x = \frac{16}{9}e^{2+t/3}$ for $t \in [t_0, 3]$.

Solve the following problems on Pontryagin maximum principle.

21. A weight on a spring follows the equation $x'' + x = u$. Find the force $u : [0, +\infty) \to [-1, 1]$ such that $x = x' = 0$ arises in the smallest time possible.

22. A police car follows the equation $x' = u$, $x(0) = 0$. Determine the motor thrust $u : [0, T] \to \mathbb{R}$ so that $P[u(\cdot)] = -\int_0^T (x(t) - z(t))^2 + \alpha u^2(t) dt$ is maximal. Time $T > 0$, the constant $\alpha > 0$ and the trajectory of the criminal $z(t)$ is given. – Solve the problem generally and then for the following particular cases (i) $z(t) = 1$, (ii) $z(t) = t$ and (iii) $z(t) = \cos t$, $\alpha = 1$, $T = 2\pi$.

23. The equation $x' = x + u$ is given. Determine $u : [0, T] \to \mathbb{R}$ such that $P[u(\cdot)] = -\int_0^T x^2(t) + u^2(t) dt$ is maximal. * Seek a control in the form of feedback, i.e. find an equation for c, where $u(t) = c(t)x(t)$.

24. Maximize $P[u(\cdot)] = \int_0^2 2x(t) - 3u(t) dt$, where $x' = x + u$, $x(0) = 4$ and $u : [0, T] \to [0, 2]$.

25. Maximize $P[u(\cdot)] = \int_0^4 3x(t) dt$, where $x' = x + u$, $x(0) = 5$ and $u : [0, T] \to [0, 2]$.

26. Maximize $P[u(\cdot)] = \int_0^2 x(t) - u^2(t) dt$, where $x' = u$, $x(0) = 0$ and $u : [0, T] \to \mathbb{R}$.

27. Maximize $P[u(\cdot)] = -\frac{1}{2} \int_0^1 x^2(t) + u^2(t) dt$, where $x' = u - x$, $x(0) = 1$ and $u : [0, T] \to \mathbb{R}$.

28. Consider the problem (interpret geometrically!)

$$\begin{aligned} x' &= \cos u, \quad y' = \sin u \\ x(0) &= y(0) = 0 \\ P[u(\cdot)] &= \max\{\sqrt{x^2(t) + y^2(t)},\ t \in [0, T]\} \end{aligned}$$

Prove that it does not attain its minimum over admissible controls in $L^\infty(0, T)$.

29. Your weekend house has a temperature $x(0) = x_0$ and you want it to be reasonably warm also at time $t = T$, not spending too much on the energy bill. A possible model is

$$\begin{aligned} x' = -kx + u, \qquad x(0) = x_0 \\ \max P[u(\cdot)] = \log x(T) - \int_0^T cu(t)\, dt \end{aligned}$$

where $u(t) : [0, T] \to [0, M]$ is the heating power, $k > 0$ is the temperature decay, and $c > 0$ is the price of energy. Try to identify optimal $u(\cdot)$.

30. Same problem as above, but with admissible controls $u(t) : [0,T] \to [0,\infty)$ and the functional

$$P[u(\cdot)] = \beta x(T) - \int_0^T u(t) + \alpha u^2(t)\, dt$$

31. * Let $P[u(\cdot)] = \int_0^T \phi(u(t))\, dt$, where ϕ is a convex, C^1 function.
(i) Prove that if $u_n \overset{*}{\rightharpoonup} u_*$ in $L^\infty(0,T)$ and $P[u_n(\cdot)] \to P_*$, then $P[u_*(\cdot)] \le P_*$.
(ii) Use this to prove existence of optimal control in previous exercise (the weekend house problem).
(iii) Show that the inequality in (i) might be strict.

Solutions.

21) Let us transform the equation to a system (14), where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The existence of an optimal control is guaranteed by Theorems 4, 5. The maximum principle gives us

$$(-h_1 \sin t + h_2 \cos t)u(t) = \max_{|\eta|\le 1}(-h_1 \sin t + h_2 \cos t)\eta,$$

for a suitable non-zero vector (h_1, h_2). We can write $(h_1, h_2) = (a \sin\omega, a\cos\omega)$, where $a > 0$, $\omega \in \mathbb{R}$. From $\max_\eta \eta a \cos(t + \omega)$; we see that the optimal u changes the value 1 and -1 with period π, more precisely

$$u(t) = \operatorname{sgn} \cos(t + \omega). \tag{26}$$

For $u = \pm 1$ the solutions are circles (in the equation (x, x')) with centers $(\pm 1, 0)$. It's easy to think through, that for each initial condition there exists a unique optimal control in the form (26).

22) The Hamiltonian $H = pu - \alpha u^2$ has only one maximum for $u = p/2\alpha$. In the optimal case we then have $x' = p/2\alpha$, $x(0) = 0$; the adjoint problem is $p' = 2x - 2z$, $p(T) = 0$. That can be transformed to a single equation

$$x'' - \frac{x}{\alpha} = -\frac{z}{\alpha}, \qquad x(0) = 0,\ x'(0) = c,$$

where c is determined so that $x'(T) = 0$ ($\iff$ $p(T) = 0$). For the specific functions z we have firstly

$$x = 1 - \cosh(t/\sqrt{\alpha}) + c\sqrt{a}\sinh(t/\sqrt{\alpha}) \tag{i}$$

from that $c = \tanh(T/\sqrt{\alpha})$. In the second case

$$x = t - \sqrt{\alpha}(c - 1)\sinh(t/\sqrt{\alpha}); \tag{ii}$$

from which $c - 1 = 1/\cosh(T/\sqrt{\alpha})$. And finally for the third case

$$x = \frac{\cos t}{2} + \frac{2c-1}{4}e^t + \frac{2c+1}{4}e^{-t}; \tag{iii}$$

from which $c = \tanh(2\pi)/2$.

23) The Hamiltonian $H = px - x^2 + pu - u^2$ has a unique maximum for $u = p/2$. The adjoint problem is in the form $p' = 2x - p$, $p(T) = 0$. We need to find the solution of the system

$$\begin{aligned} x' &= x + p/2, \qquad & x(0) &= x_0, \\ p' &= 2x - p, \qquad & p(0) &= p_0, \end{aligned}$$

where p_0 is chosen so that $p(T) = 0$. Feedback: set $d(t) = p(t)/x(t)$ and thus $c(t) = d(t)/2$. Following the substitution, the equation for $d(t)$ is

$$d' = 2 - 2d - \frac{d^2}{2}, \qquad d(T) = 0.$$

Finally, its solution can be found it the form of $d(t) = 2b'(t)/b(t)$, where the auxiliary function $b(t)$ satisfies the equation

$$b'' + 2b' - b = 0, \qquad b(T) = 1, \ b'(T) = 0,$$

which we know how to solve.

24) $u = 2$, $x = 6e^t - 2$ on $[0, t_0]$; $u = 0$, $x = (6 - 2e^{-t_0})e^t$ on $[t_0, 2]$, where $t_0 = 2 - \ln(5/2)$.

25) $u = 2$, $x = 7e^{-t} - 2$ on $[0, 4]$.

26) $u = -t/2 + 1$, $x = -t^2/4 + t$ on $[0, 2]$.

27) $u = c_1(\sqrt{2} + 1)e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$, $x = -c_2(\sqrt{2} + 1)e^{-\sqrt{2}t} + c_1 e^{\sqrt{2}t}$, where c_1, c_2 is such that $x(0) = 1$ and $u(1) = 0$.

28) The infimum is zero: take suitable piecewise constant u; alternatively, take $u = Nt$ for very large N. However, $P = 0$ would mean $x = y = 0$ for all t, which is not possible.

29) Adjoint equation $p' = kp$, $p(T) = 1/x(T)$; at most one change from $u = 0$ to $u = M$.

30) Adjoint equation $p' = kp$, $p(T) = \beta$ can be solved explicitly; control $u(t)$ can be expressed in terms of $p(t)$ (minimization of a quadratic function – however, beware of the condition $u \geq 0$.)

31) (i) By convexity $\phi(v) \geq \phi(u) + \phi'(v)(u - v)$ for all u, $v \in \mathbb{R}$. Set $v = u_n(t)$, $u = u_*(t)$, integrate $\int_0^T dt$ and ...
(iii) $u_n(t) = \cos(nt) \overset{*}{\rightharpoonup} 0$ (by Riemann-Lebesgue), but $(u_n(t))^2 \overset{*}{\rightharpoonup} 1/2$ by the formula $\cos^2 y = (1 + \cos 2y)/2$