Dynamical systems

Definition 1. A dynamical system is defined as a pair (φ, Ω), where $\Omega \subset \mathbb{R}^n$ and $\varphi(t,x) : \mathbb{R} \times \Omega \to \Omega$ is a continuous mapping satisfying "the semigroup property":

(i) $\varphi(0,x) = x$ for all $x \in \Omega$

(ii) $\varphi(s, \varphi(t,x)) = \varphi(s+t,x)$ for all $x \in \Omega$, t, $s \in \mathbb{R}$

Canonic example. The differential equation

$$x' = f(x)\,, \tag{1}$$

where $f : \Omega \to \mathbb{R}^n$ is a given function, determines a dynamical system (φ, Ω) by a solving function

$$\varphi : (t, x_0) \mapsto x(t)\,, \tag{2}$$

where $x(t)$ is a solution of (1) with the initial condition $x(0) = x_0$. From general theorems about existence and uniqueness we get that if $f \in C^k$, $k \geq 1$ then φ is correctly defined (at least for t close to 0) and φ is also of class C^k.

Remark. Conversely, it is possible to show that any dynamical system (if $\varphi \in C^1$) is a solving function of a certain differential equation of the form (1). See exercise 4 below. While studying this theory it is therefore expedient to bear in mind the unique correspondence equation $\iff$ dynamical system.

Let's additionaly note that particularly for linear equations

$$x' = Ax$$

where A is a constant matrix, it is possible to write the corresponding system explicitly using the exponential of the matrix:

$$\varphi(t,x) = e^{tA}x = \sum_{k=0}^{\infty} \frac{t^k A^k x}{k!}\,.$$

Remark. For a given equation, it is generally, i.e. for a non-linear f, a non-trivial problem to determine whether its corresponding dynamical system is defined for all $t \in \mathbb{R}$ (i.e. to answer the question about the global existence of the solution). Simple examples show that the solution may go to infinity (a so called ,,blow-up“) in finite time. The following criterion will suffice for most reasonable applications.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, let the solutions of the equation* (1) *with their corresponding initial conditions in Ω be such that they can not leave Ω. Then these solutions are defined for all $t \in \mathbb{R}$.*

How to practically ensure that the solutionos will not leave Ω ?

1. the vector $f(x)$ on the boundary is oriented strictly to the inside of Ω – since the solutions are the curves $x = x(t)$, whose tangent is $f(x(t))$, it is impossible to get to the boundary from ,,the inside“.
2. the boundary $\partial\Omega$ is itself comprised of solutions and equilibriums – in this case it is also impossible to cross the boundary (it would be a contradiction to the uniqueness of the solution).

Definition 2. Let (φ, Ω) be a dynamical system. The set M is called *invariant*, if $x \in M$ implies $\varphi(t, x) \in M$ for all $t \in \mathbb{R}$.

The key object for studying the behaviour of dynamical systems for large times is contained in the following definition.

Definition 3. Let (φ, Ω) be a dynamical system, let $x_0 \in \Omega$. The omega-limit set of the point x_0 is defined as follows:

$$\omega(x_0) = \big\{ y \in \Omega; \quad \text{there exist } t_k \to \infty \text{ such that } \varphi(t_k, x_0) \to y \ \big\} .$$

Analogously we define the alfa-limit set:

$$\alpha(x_0) = \big\{ y \in \Omega; \quad \text{there exist } t_k \to -\infty \text{ such that } \varphi(t_k, x_0) \to y \ \big\} .$$

Theorem 2. *The set $\omega(x_0)$ is closed and invariant. If in addition the positive trajectory*

$$\gamma^+(x_0) = \big\{ \varphi(t, x_0); \ t \geq 0 \big\}$$

is relatively compact, then $\omega(x_0)$ is non-empty, compact and connected.

Remark. The omega-limit set is comprised of all accumulation points of the positive trajectory; more precisely it is comprised of all points in whose arbitrarily small neighbourhood we will find the solution for arbitrarily large times.
The condition of the relative compactness (in case of $\Omega \subset \mathbb{R}^n$ it is equivalent to the boundedness of the corresponding orbit) in the second part of the theorem is essential: see exercises 2, 3 below.

Exercises on dynamical systems

1. Find the explicit solving function $\varphi(t, x)$ for the following equations/systems

(i) $x' = x^p$, $p \in \mathbb{R}$ (pro $x > 0$)

(ii) $x'' + x = 0$ (rewrite as a system of 2 equations)

(iii) $x' = x + \ln y$, $y' = -y$ (for $y > 0$)

(iv) $x' = y^2 - x^2$, $y' = -2xy$ (use the complex form $z = x + iy$)

Check that (at least locally) the following property of a dynamical system holds: $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$.

2. Find a dynamical system in $\mathbb{R}^2$ for a suitable initial point x_0:

(i) $\omega(x_0) = \emptyset$

(ii) $\omega(x_0)$ is a unit circle

(iii) $\omega(x_0)$ is comprised of two points

(iv) $\omega(x_0)$ is a straight line

(v) $\omega(x_0)$ is a unit disc

3. (Non-connected ω-limit set) Consider the system of equations

$$\begin{aligned} x' &= -y(1 - x^2), \\ y' &= x + y(1 - x^2). \end{aligned}$$

Let us restrict ourselves to a vertical strip $|x| < 1$, and:
(i) find and anaylze equilibria
(ii) identify curves in which x' or y' changes the sign and sketch the trajectories of the solutions
(iii) show that for any point $x_0 \neq 0$ the set $\omega(x_0)$ is equal to the union of lines $x = \pm 1$

4. Let $\varphi(t, x)$ be a smooth dynamical system in $\mathbb{R}$. Show that for any fixed x_0 the function $x(t) := \varphi(t, x_0)$ is a solution of the equation $x' = f(x)$, where $f(\xi) := \frac{\partial \varphi}{\partial t}(0, \xi)$.

5. Proof the following theorem: let us have a dynamical system in $\mathbb{R}^n$. Let $\omega(x_0) = \{z\}$. Then $\lim_{t\to\infty} \varphi(t, x_0) = z$. (In terms of differential equations: the solution starting at the point x_0 converge to z for $t \to \infty$.)

6. Let (φ, Ω) be a dynamical system; let $\gamma^+(x_0)$ be relatively compact. Then $\varphi(t, x_0) \to \omega(x_0)$ for $t \to \infty$ in the sense of distance[1] of sets, i.e.

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, x_0), \omega(x_0)) = 0.$$

7. Find a dynamical system in $\mathbb{R}^2$ (or show, that it does not exist), such that $\omega(x_0)$ is a unit circle

[1] Set $\operatorname{dist}(a, M) = \inf_{x\in M} |a - x|$.

(i) for every $x_0 \in \mathbb{R}^2 \setminus \{(0,0)\}$

(ii) for every $x_0 \in \mathbb{R}$

8. Prove the following. Let $\omega(x_0)$ be disconnected. Then bounded components of $\omega(x_0)$ are not isolated (i.e. every bounded component has a zero distance from another component).

Solutions

1) (i) if $p = 1$, then $x_0 \mapsto x_0 e^t$, $t \in \mathbb{R}$; for $p \neq 1$ we have $x_0 \mapsto [t(1-p) + x_0^{1-p}]^{\frac{1}{1-p}}$, with the restriction $t > -\frac{x_0^{1-p}}{1-p}$, if $p < 1$, and with the restriction $t > -\frac{x_0^{1-p}}{1-p}$, if $p > 1$.
(ii) $(x_0, y_0) \mapsto (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t)$; consider, that $\varphi(t, \cdot)$ is linear
(iii) $(x_0, y_0) \mapsto (x_0 e^t + (\ln y_0 - t)(e^t - 1), y_0 e^{-t})$
(iv) the equation has a general solution $z(t) = z_0/(1 + t z_0)$, the orbits are circles with centers on the imaginary axis.

2) (i) $\varphi(t, x) = t + x$, $t \in \mathbb{R}$;
(ii) The system generated by the following equations (in polar coordinates) $r' = r - r^2$, $\phi' = -1$; the solutions are the circle $r = 1$ and the origin (which is the equilibrium). Every other solution is a spiral winding up to the unit circle which is the ω-limit set. In cartesian coordinates the system looks as follows

$$\begin{aligned} x' &= x + y - x\sqrt{x^2 + y^2}, \\ y' &= -x + y - y\sqrt{x^2 + y^2}. \end{aligned}$$

(iii) has no solution – if $\omega(x_0) = \{a, b\}$, then choose $\delta > 0$ such that $U(a, 2\delta) \cap U(b, 2\delta) = \emptyset$. However the orbit of the point x_0 intersects the circle $\{|x| = \delta\}$ for any arbitrarily large times; from compactness there exists another element $\omega(x_0)$, which is a contradiction;
(iv) use the case (ii) and the conformal mapping of the plane, which transforms circles into straight lines;
(v) has no solution, $\omega(x_0)$ must always have an empty interior.

3) (i) $(0, 0)$ – unstable vortex, from which spirals run out of counterclockwise;
(ii) $x' = 0$ for $y = 0$, $y' = 0$ for $y = x/(x^2 - 1)$, which indicates the same spiral trajectory;
(iii) from the Bendixson-Dulac criterion (choose $B = 1/(x^2 - 1)$) we get nonexistence of periodic solutions inside the strip. Thus the sequence ξ_k of points of intersection of a (nontrivial) solution with the line segment $\{(x, 1);\ 0 < x < 1\}$ is injective (increasing) and can not have an accumulation point *inside*

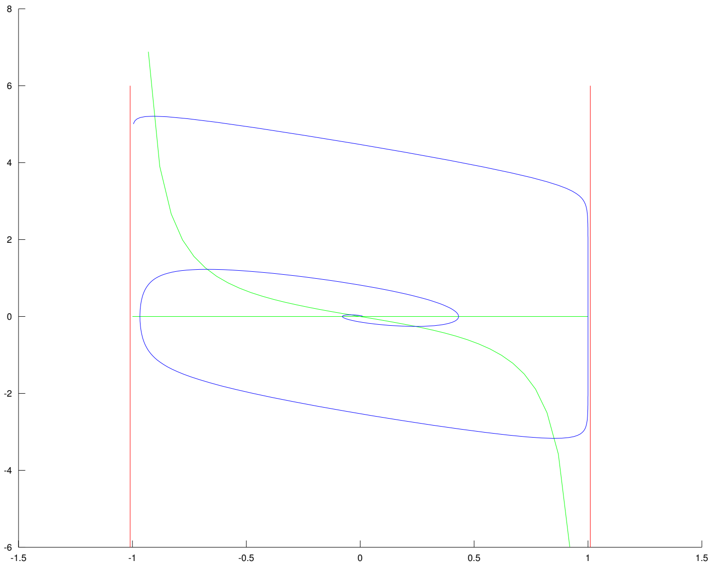

Figure 1: Úloha 3

the strip (since it would have to lie on a periodical orbit, which we know from the proof of the Poincaré-Bendixson theorem). Necessarily $\xi_k \to 1-$ and therefore the spiral gets „locally uniformly“ close to the line $x = 1$. The symetrical argument holds on the other side. — See Figure 1 (solution in blue, izolines in green, omega-limit set in red).

4) Show that for any t_0 fixed it holds

$$x(t_0 + t) = \varphi(t_0 + t, x_0) = \varphi(t, \varphi(t_0, x_0)).$$

From this (by differentiation at $t = 0$) we get

$$x'(t_0) = \frac{\partial \varphi}{\partial t}(0, \varphi(t_0, x_0)) = f(\varphi(t_0, x_0)) = f(x(t_0)).$$

5) If z is not the limit, then the orbit $\varphi(t, x_0)$ is situated outside of some ε-neighbourhood for arbitrarily large times. At the same time (since $z \in \omega(x_0)$) it is situated in this neighbourhood for arbitrarily large times. Therefore there exists a sequence $t_k \to +\infty$, such that $|\varphi(t_k, x_0) - z| = \varepsilon$. Then there would exist another element in $\omega(x_0)$ on the (compact) sphere $\{x \in \mathbb{R}^n;\ |x - z| = \varepsilon\}$ – a contradiction.

6) If the conclusion does not hold, there exists a $\delta > 0$ and a sequence $t_k \to \infty$ such that $\operatorname{dist}(\varphi(t_k, x_0), \omega(x_0)) \geq \delta$. By the assuption of compactness $\varphi(t_k, x_0)$ has an accumulation point which is an element of $\omega(x_0)$ – a contradiction.

7) (i) The orbits create a system of spirals originating in the origin or in infinity, all of them converging to the unit circle.
(ii) This is impossible, $\alpha(x_0)$ for $|x_0| < 1$ would also have to be a part of the unit circle, and thus it would have to be the whole circle by the Poincaré-Bendixson theorem. A small line segment intersecting the circle perpendicularly is a transversal, the intersections with the orbit must create a monotone sequence on it, which is a contradiction with the fact that the circle is an α and ω-limit set.

8) Consider a bounded isolated component K and $U_\varepsilon := \{x \in \mathbb{R}^n : d(x, K) < \varepsilon\}$. If $\gamma_+(x_0)$ for $t \geq t_0$ stays in U_ε, we have a contradiction with $\omega(x_0)$ being disconnected. If the orbit will still leave the set U_ε, it will have an accumulation point in $\overline{U_\delta} \setminus U_{\delta/2}$, namely for every $\varepsilon > \delta > 0$. That is a contradiction with K being isolated.

La Salle's invariance principle

The first application of the theory of dynamical systems, which uses the term ω-limit set, is the so called La Salle's invariance principle.

Motivational example. Let us consider the equation

$$x'' = -x - q(x'),$$

which describes a pendulum with friction, i.e. acceleration is equal to minus deviation minus the friction force. A reasonable assumption on the friction force is

$$q(0) = 0, \qquad q(y)y > 0 \quad \text{pro } y \neq 0, \tag{3}$$

since friction works strictly against the direction of movement.¨ Based on the physical intuition we expect, that the rest state $x(0) = x'(0) = 0$ is asymptotically stable.
Let us rewrite the equation as a system in $\mathbb{R}^2$:

$$\begin{aligned} x' &= y, \\ y' &= -x - q(y). \end{aligned}$$

The linearization in the origin leads to the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix},$$

where $a = -q'(0)$. From the assumption (3) it holds that $q'(0) \geq 0$, which generally does not tell us anything (for $a = 0$ the spectrum A is equal to $\{i, -i\}$.) A stronger assumption $q'(0) > 0$ guarantees that the origin is asymptotically stable.

It seems handy to use the the Lyapunov function $V = x^2 + y^2$. The orbital derivative is

$$\dot{V} = 2xx' + 2yy' = 2xy - 2xy - 2yq(y) = -2yq(y) \leq 0.$$

Therefore supposing (3) the origin is stable; the Lyapunov theorem is however insufficient for the asymptotic stability, because $\dot{V} < 0$ only for $y \neq 0$, not – as we would need – for all $(x, y) \neq (0, 0)$.
Is it possible to make this theorem more precise? In reality $\dot{V} < 0$,,almost always", the solutions actually orbit around the origin on a spiral and the ,,unpleasant" set $y = 0$ gets intersected only sometimes. It is therefore natural to suppose, that $V \to 0$, i.e. the system heads towards the origin for $t \to \infty$.
This is formulated precisely in the following theorem. Let us remind that $\omega(x_0)$ denotes the omega-limit set of a given point and $\gamma(x_0)$ denotes the (complete) orbit, which comes out of the point, i.e.

$$\gamma(x_0) = \big\{x(t),\ t \in \mathbb{R};\ x(t) \text{ is a solution with the initial condition } x(0) = x_0\big\}$$

Theorem 3. *[La Salle's invariance principle.] The equation $x' = f(x)$ is given, where $f(x) : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$.*
Let there exist a function $V(x) : \Omega \to \mathbb{R}$, which is of class C^1, bounded from below, and let there exist an $\ell \in \mathbb{R}$ such that, the set

$$\Omega_\ell := \big\{x \in \Omega;\ V(x) < \ell\big\}$$

is bounded and

$$\dot{V}(x) \leq 0 \quad \forall x \in \Omega_\ell.$$

Let us now set

$$\begin{aligned} R :=& \big\{x \in \Omega_\ell;\ \dot{V}(x) = 0\big\}, \\ M :=& \big\{x_0 \in R;\ \gamma(x_0) \subset R\big\}. \end{aligned}$$

Then for any $x_0 \in \Omega_\ell$ we have $\omega(x_0) \subset M$.

Remark. The theorem assumptions guarantee that the set Ω_ℓ is positively invariant, i.e. the solutions can not leave it with increasing time. Equivalently the set M is the largest invariant subset of R. The conclusion of the theorem

– $\omega(x_0) \subset M$ – says that the orbits originating at $x_0 \in \Omega_\ell$ have accumulation points only inside M; considering the compactness (boudness of Ω_ℓ) we already necessarily have, that the orbits approach M in the sense of distance of sets. Specially, if M contains a single point, the solutions converge to this point. (Compare exercises 5, 6).

Example – completion. Choose $V = x^2 + y^2$, $\Omega = \mathbb{R}^2$, $\ell > 0$ arbitrarily. Thus

$$R = \big\{(x, y) \in \Omega_\ell;\ \dot V = 0\big\} = \big\{(x, 0);\ -\ell < x < \ell\big\}$$

However, if $(x, 0) \in R$, $x \neq 0$, from the equation we have $y' = -x \neq 0$, i.e. the solutions immediately leaves the set R. The only invariant subset of R is therefore $M = \{(0, 0)\}$.
From the theorem above we have that the ω-limit set of every point from $\Omega_\ell = \{(x, y);\ x^2+y^2 \leq \ell^2\}$ is the origin $(0, 0)$, i.e. the origin is asymptotically stable.

Exercises on the La Salle's invariance principle

9. Let f, $g \in C^1(\mathbb{R})$ be increasing, $f(0) = g(0) = 0$. Examine the stability of the equilibriums of the system

$$\begin{aligned} x' &= -y - f(x), \\ y' &= g(x). \end{aligned}$$

Hint: $V = \int g(x)dx + y^2/2$.

10. Find ω-limit sets for the system

$$\begin{aligned} &x' = y - x^7(x^4 + 2y^2 - 10), \\ &y' = -x^3 - 3y^5(x^4 + 2y^2 - 10). \end{aligned}$$

Hint: $V = (x^4 + 2y^2 - 10)^2$.

11. Show that the origin is globally asymptotically stable for the system

$$x' = -y - x^3, \qquad y' = x^5.$$

Hint: $V = x^n + y^m$ *for suitable even* m, n.

12. Show that the origin is globally asymptotically stable for the system

$$x' = -x^3 + 2y^3, \qquad y' = -2xy^2.$$

Solutions

9) The origin is the only equilibrium. (The linearization gives us the asymptotic stability, if additionaly $f'(0) > 0$ and $g'(0) > 0$.) Set $G(x) = \int_0^x g(\xi)d\xi$ and define $V = G(x) + y^2/2$. Then (by the setting of Theorem 3) we have that Ω_ℓ is always bounded; $\dot{V} = -f(x)g(x)$, thus $R = \Omega_\ell \cap \{x = 0\}$ and since $x' \neq 0$ for $x = 0$, $y \neq 0$, we have that M is equal to the origin, which is therefore globally asymptotically stable.

10) Since $\dot{V} = -8V(x^2 + 2y^6) \leq 0$, it holds that $M = R = \{0\} \cup E$, where E is "the ellipse" $\{x^4 + 2y^2 = 10\}$. If x_0 is different from the origin, then $0 \notin \omega(x_0)$, (since V has a local maximum in 0); and thus $\omega(x_0) \subset E$. Moreover necessarily $\omega(x_0) = E$ and E is a periodical orbit (by elementary considerations or by Poincaré-Bendixson theorem).

11) Choose $V = x^6/3 + y^2$, and thus $\dot{V} = -2x^8$. Moreover it holds $R = \Omega_\ell \cap \{x = 0\}$, however M is only the origin, which is therefore (globally) asymptotically stable thanks to the exercise 5.

12) $V = x^2 + y^2$, $\dot{V} = -4x^4$; then analogously to the exercise 11.

Poincaré-Bendixson theory

During this chapter we will consider the dynamical system (φ, Ω), where Ω is a region (i.e. an open connected set) in $\mathbb{R}^2$. The function $\varphi = \varphi(t, x)$ is defined at least for every $t \geq 0$, $x \in \Omega$ and it is continuously differentiable. Let us remark, that the restriction on the two-dimensional dynamics in the whole theory is *crucial*, since it is related to the topology of the plane.
Let us remind some terms: *a simple closed curve* is a set $\gamma \subset \Omega$ such that $\gamma = \phi([0,1])$, where $\phi : [0,1] \to \Omega$ is a continuous mapping, which is injective on $[0,1)$ and it holds that $\phi(0) = \phi(1)$. Obviously the orbit of a (non-trivial) periodical solution is a simple closed curve. The Jordan theorem holds in the plane: if γ is a simple closed curve then we can (disjunctively) write

$$\mathbb{R}^2 = M_1 \cup \gamma \cup M_2,$$

where M_i are open connected sets; moreover M_1 is bounded and M_2 is unbounded. It is useful to set $M_1 = \operatorname{int} \gamma$.
The main result is the following theorem.

Theorem 4 (Poincaré-Bendixson)**.** *Let $p \in \Omega$ be such that $\overline{\gamma^+(p)}$ is compact and let $\omega(p)$ not contain an equilibrium. Then $\omega(p) = \Gamma$, where Γ is the orbit of a non-trivial periodic solution.*

Remark. Compactness of the forward orbit follows from (in fact is is equivalent to) its boundedness; specially it is sufficient to suppose, that Ω is bounded.

We can usually exclude the existence of equilibria in $\omega(p)$ by considerations similar to Exercise 15 below; notice however, that considering the assumption of the theorem, the set Ω always contains equilibria – see Exercise 14.

Example 1. Show that the system

$$\begin{aligned} x' &= x - y - x^3 \\ y' &= x + y - y^3 \end{aligned}$$

has a non-trivial periodic solution.

Solution. The qualitative analysis shows that we have only one equilibrium $(0, 0)$. The sings of the derivatives outside of "isoclines" $x' = 0$, $y' = 0$ respectively force the solution to a (clockwise) spiral-like movement around the origin.
The linearization in the point $(0, 0)$ is determined by the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues $1 \pm i$. From this we further get (see Exercise 15), that $(0, 0)$ is not a part of the ω-limit set of any orbit.

We will now design a forward-invariant set Ω using the Lyapunov function $V = x^2 + y^2$. The orbital derivative is

$$\begin{aligned} \dot{V} &= 2xx' + 2yy' \\ &= 2x(x - y - x^3) + 2y(x + y - y^3) \\ &= 2(x^2 + y^2 - (x^4 + y^4)). \end{aligned}$$

We can easily check that $\dot{V} < 0$ if $x^2 + y^2 = R^2$, where $R > 0$ is a number large enough. Therefore, the set

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2;\ x^2 + y^2 < R^2 \right\}$$

is positively invariant; the dynamical system (φ, Ω) is defined of all $t \geq 0$ and its orbits are compact. From Theorem 4 we now get that every solution in Ω converges as $t \to \infty$ to the periodical orbit Γ.

The Poincaré-Bendixson theorem specially guarantees *the existence* of a non-trivial periodic solution. The following theorem on the contrary contains a useful negative criterion.
Let us remind, that the region $\Omega \subset \mathbb{R}^2$ is called simply connected, if for any simple closed curve $\gamma \subset \Omega$ it holds that $\operatorname{int} \gamma \subset \Omega$. Equivalently: every simple closed curve can be continuously contracted into a point without leaving Ω.

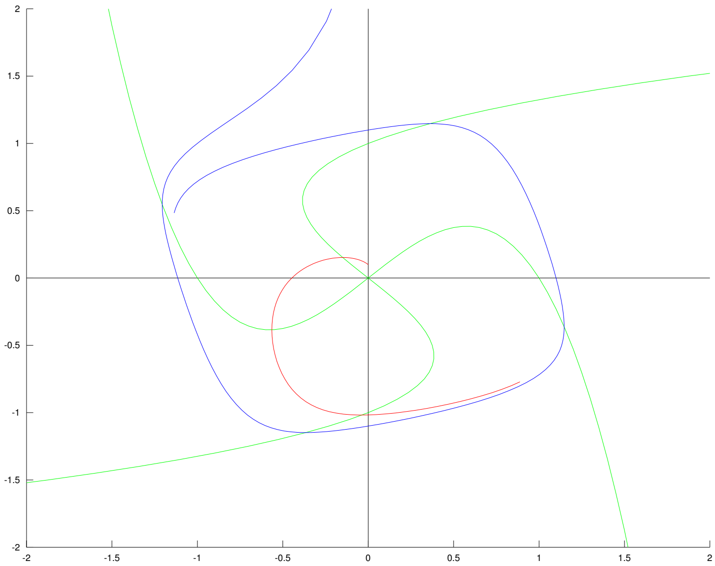

Figure 2: Exercise 1

Theorem 5 (Bendixson-Dulac). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected region; let $f : \Omega \to \mathbb{R}^2$ be a function of class C^1 and let there exist a function $B : \Omega \to \mathbb{R}$ of class C^1 such that $\operatorname{div}(Bf) > 0$ almost everywhere in Ω. Then the equation $x' = f(x)$ has no (non-trivial) periodic solutions in Ω.*

Exercises on Poincaré-Bendixson theory

13. Examine the existence of periodic solutions for the following system by transforming it into polar coordinates:

$$x' = ax - y + xy^2,$$
$$y' = x + ay + y^3$$

depending on the parameter $a \in \mathbb{R}$.

14. Let $\Omega \subset \mathbb{R}^2$ be a simply connected region; let $\Gamma \subset \Omega$ be a periodical orbit. Then $\operatorname{int} \Gamma$ contains at least one equilibrium. *Hint: by the Zorn lemma, there exists a smallest compact invariant subset of the interior of Γ. Show that it necessarily contains only a single point.*

15. Let x_0 be an equilibrium of the equation $x' = f(x)$, let $A = \nabla f(x_0)$ have all eigenvalues with a positive real part. Then x_0 is not contained in the ω-limit set of any point (except x_0).

16. Show that the van der Pol equation

$$x'' + x'(x^2 - 1) + x = 0$$

has a periodical solution.

17. Show that the system

$$x' = -y + x(1 - x^2 - 2y^2),$$
$$y' = x + y(1 - 2x^2 - y^2)$$

has a periodical solution.

18. Show that the system

$$x' = 1 - xy, \qquad y' = x$$

has no periodical solutions.

19. Show that the system

$$x' = x + xy^2, \qquad y' = (1 - y^2)/2$$

has no periodical solutions

20. Show that the system

$$x' = \frac{x - 2y}{1 + x^2 + y^2}, \qquad y' = \frac{2x - y/2}{1 + x^2 + y^2}$$

has no periodical solutions.

Solutions

13) $r' = r(a + r^2 \sin^2 \phi)$, $\phi' = 1$. From this $\phi = t + c$ and therefore we are searching for the 2π-periodic solutions of the equation for r. For $a \geq 0$ there does not exist such a solution (moreover $r \to \infty$ in finite time); for $a < 0$ there does (the equation for r is Bernoulli and it is possible to show, that it has a unique 2π-periodic solution).

14) Assume there is no equilibrium. The set

$$\mathcal{K} = \big\{K \subset \operatorname{int} \Gamma;\ K \text{ is compact and invariant } \big\}$$

contains, thanks to the Zorn lemma, a smallest element (in the sense of inclusion). The non-emptiness of $\mathcal{K}$ (just as the fact, that the minimal element is necessarily a single point) follows from this observation: if K is invariant and $p \in \operatorname{int} K$, then $\gamma_1 = \omega(p)$, $\gamma_2 = \alpha(p)$ are periodic orbits by Theorem 4; since $\gamma_1 \neq \gamma_2$ (e.g. from the lemma about monotone intersections with the transversal), one of them together with its interior form a strictly smaller invariant set.

15) WLOG $x_0 = 0$. Let $x(t)$ be a non-trivial orbit such that $x(t_k) \to 0$, where $t_k \to \infty$. Using the theorem about linearized stability on the equation with the time reversed, we get that 0 is negatively stable:

$$(\forall \varepsilon > 0)(\exists \delta > 0) \quad \big[|x(\tau)| < \delta \implies |x(t)| < \varepsilon \qquad \forall t \leq \tau\big].$$

Let us choose $0 < \varepsilon < |y(t_1)|$ and for the obtained $\delta > 0$ let us find $t_k > t_1$ such that $|y(t_k)| < \delta$. That is a contradiction.

16) Rewrite the system for $(x, y) = (x, x')$. The origin is the only equilibrium and it is negatively stable (see Exercise 15). The qualitative analysis indicates a clockwise movement on spirals. For $V = x^2 + y^2$ it holds $\dot{V} \leq 0$ if

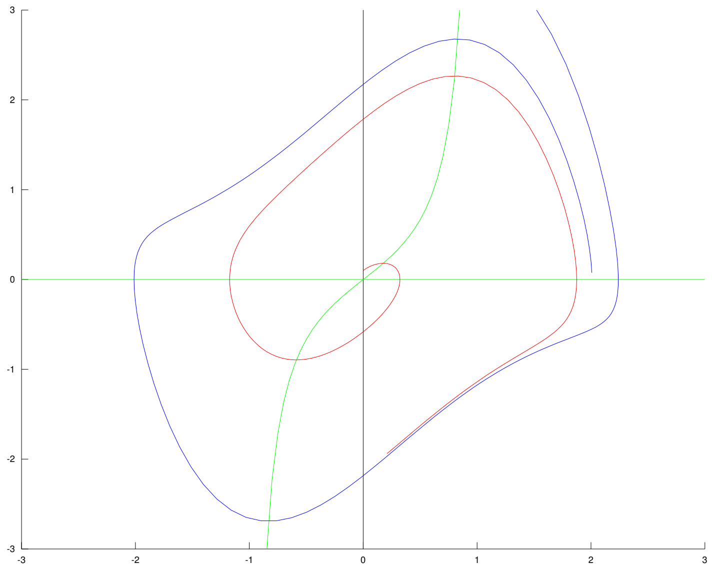

Figure 3: Exercise 16

$|x| \geq 1$; we therefore bound the positively invariant Ω on the sides by arcs $\big\{|x| > 1\big\} \cap \big\{x^2 + y^2 = R^2\big\}$ and above and below by the solutions connecting the lines $x = \pm 1$.

17) The origin is the only equilibrium which is negatively stable. If $V = x^2 + y^2$, then $\dot{V} < 0$ for $x^2 + y^2 = R^2$, where $R > 0$ is large enough, from which we get a positively invariant ball.

18) By elementary considerations or the fact, that there exist no equlibria and Exercise 14.

19) The Dulac function $B = 1/(1 + y^2)$.

20) The Dulac function $B = 1 + x^2 + y^2$.