Center manifolds

An equilibrium $X_0 \in \mathbb{R}^N$ of the equation

$$X' = F(X) \tag{0}$$

is given. Let F be at least of class C^1 on a neighbourhood of X_0. Set $M = \nabla F(X_0)$. Let us assume that $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma(M)$ and that there exists a $\hat{\lambda} \in \sigma(M)$ such that $\operatorname{Re}\hat{\lambda} = 0$. This is the only situation in which we can not decide about the stability of X_0 on the basis of linarization theorems. More precisely, there is no general connection between the stability of the point X_0 and the stability of the origin of $Y' = MY$.

In the presented situation the linear terms are insufficient to determine the (in)stability of the point X_0. In this chapter we will show that by excluding the uninteresting case of stable dynamics (which corresponds to the eigenvectors satisfying $\operatorname{Re}\lambda < 0$), the behaviour of the equation on a neighbourhood of X_0 can be reduced to a so called *center manifold* (c.m.). We will see that the dynamics of the center manifold can be approximated with arbitrary accuracy and we will eventually solve the former problem of stability of the point X_0.

Remark. Proofs of the following theorems can be found in chapter 2 of the book *J. Carr: Applications of centre manifold theory, Springer 1981*, from which we have also borrowed some of our exercises.

Suppose that $X_0 = 0$ and that the system (0) is transformed by a suitable change of variables into

$$\begin{aligned} x' &= Ax + f(x, y) \\ y' &= By + g(x, y) \end{aligned} \tag{1}$$

where $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$\begin{gathered} \operatorname{Re}\sigma(A) = 0 \\ \operatorname{Re}\sigma(B) < -\beta < 0 \\ f(0,0) = g(0,0) = 0 \\ \nabla f(0,0) = \nabla g(0,0) = 0 \end{gathered} \tag{P}$$

The vectors x, y are called center variable and stable variable respectively.

Definition 1. A smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies $\phi(0) = \nabla\phi(0) = 0$ is called *center manifold* of the system (1), if the following holds: there exists $\mathcal{U}$ a neighbourhood of the point $(0, 0)$ such that if $(x(t), y(t))$ is a solution to (1) then the following holds:

$$y(0) = \phi(x(0)) \implies y(t) = \phi(x(t)) \quad \forall t \text{ such that } (x(t), y(t)) \in \mathcal{U}. \tag{INV}$$

The condition (INV) will be termed *the invariance property*; it says that the graph of ϕ (more precisely the set $\{(x,y);\ y=\phi(x)\}\cap\mathcal{U}$) is invariant with respect to the solutions of (1).

The invariance property is equivalent to *the reduction property*: if p is a solution to the ,,reduced equation"

$$p' = Ap + f(p,\phi(p)), \tag{2}$$

then $(x(t),y(t)) := (p(t),\phi(p(t)))$ is a solution to the original system (1) for all t that satisfy $(p(t),\phi(p(t)))\in\mathcal{U}$. The non-trivial requirement of course is to satisfy the *second* equation in (1).

Theorem 1. *Let the assumptions* (P) *hold, and let the functions f, g be of class at least C^2 on a neighbourhood of* $(0,0)$*. Then there exists a center manifold of the system* (1) *and it is also of class C^2 on a neighbourhood of* 0.

Let us remark, that the center manifold is not uniquely determined (see Exercise 1 below.) If f, g are of class C^k, it is possible to have a c.m. also of class C^k, despite the fact that the neighbourhood gets smaller when k increases: generally there need not exist an analytic c.m. (Exercise 2), not even a c.m. that is C^∞ on a neighbourhood of the origin.

The c.m. can be understood as follows: it allows to solve both parts of the system (1) separately. The stable part y has an uninteresting dynamic and its action on the central part x can be expressed by the functional relation $y=\phi(x)$. Therefore it is sufficient to solve the reduced equation (2), which contains all the important information about the behaviour of the whole system, on a neighbourhood of the origin.

This fact is expressed more precisely in the following lemma.

Lemma 2. *Let* 0 *be an equilibrium of the equation* (2)*. Then for any solution* $(x(t),y(t))$ *of the equation* (1) *with an initial condition* $(x(0),y(0))$ *small enough, there exists a solution $p(t)$ of the equation* (2) *such that*

$$\begin{aligned} x(t) &= p(t) + \mathcal{O}(e^{-\gamma t}) \\ y(t) &= \phi(p(t)) + \mathcal{O}(e^{-\gamma t}) \end{aligned}$$

for $t\to\infty$, where $\gamma>0$ is a suitable constant.

In other words: if we suppose the stability of the reduced equation, then every solution of the original equation can be approximated with an exponentially small error by a solution that lies on the c.m. We also say, that the c.m. has ,,the tracking property".

An easy corollary of the previous lemma is ,,the principle of reduced stability".

Theorem 3. *Let ϕ be a c.m. of the system* (1). *The point* $(0,0)$ *is stable (asymptotically stable, unstable) for* (1), *if and only if the point* 0 *has the analogous property for* (2).

Center manifold approximation

Let's return to the original problem: the stability of the point $(0,0)$ for the system (1). The theorem at the end of the previous section solves that question only theoretically: practically it does not help us if we can not find the center manifold ϕ.

By the reduction principle ϕ is a center manifold if and only if $(x(t), y(t)) := (p(t), \phi(p(t)))$ is a solution of (1), whenever p solves (2). The equation $(1)_2$ requires the following:

$$\begin{aligned}\big(\phi(p)\big)' = \nabla\phi(p)p' = B\phi(p) + g(p, \phi(p))\\ \nabla\phi(p)\big(Ap + f(p, \phi(p))\big) = B\phi(p) + g(p, \phi(p))\end{aligned}$$

We see that ϕ is a center manifold if and only if the last identity holds for every solution of the equation (2) on a neighbourhood of 0. Clearly, this is the case if and only if $M[\phi] \equiv 0$ on a neighbourhood of 0, where

$$M[\phi](x) = \nabla\phi(x)\big(Ax + f(x, \phi(x))\big) - B\phi(x) - g(x, \phi(x)). \tag{3}$$

Generally we do not know how to solve this partial differential equation. From a practical point of view, the following approximation theorem is satisfactory.

Theorem 4. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^m$ be a class C^1 function satisfying $\psi(0) = \nabla\psi(0) = 0$. Let $M[\psi](x) = \mathcal{O}(|x|^q)$, $x \to 0$ for some $q > 1$. Then there exists a center manifold ϕ satisfying $\phi(x) - \psi(x) = \mathcal{O}(|x|^q)$, $x \to 0$.*

Example 1. Examine the stability of the origin of the following system

$$\begin{aligned}x' &= -x^3 + y^2,\\ y' &= -2y + x^2.\end{aligned}$$

Solution. The given system is of the form (2), where $m = n = 1$, $A = 0$, $B = -2$, $f = -x^3 + y^2$ and $g = x^2$. By Theorem 1 we want to find a center manifold of the form $y = \phi(x)$. The expression to approximate (3) is

$$M[\phi](x) = \phi'(x)\big(-x^3 + \phi^2(x)\big) - \big(-2\phi(x) + x^2\big).$$

Let us set $\psi(x) = 0$. Then $M[\psi] = -x^2$; by Theorem 4 the center manifold is in the form $\phi(x) = 0 + \mathcal{O}(x^2)$. The reduced equation is therefore in the form

$$p' = -p^3 + \big(0 + \mathcal{O}(p^2)\big)^2 = -p^3 + \mathcal{O}(p^4).$$

This equation is asymptotically stable at zero (see Exercise 4); the same holds for the origin of the original system.

Example 2. Examine the stability of the origin of the system

$$\begin{aligned} x' &= x^2y^3 \\ y' &= -2y - y^3 + ax^3 \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter.

Solution. Once again we see that x is the center variable, y is the stable variable. The expression for approximation has the form

$$M[\psi](x) = \psi'(x)x^2\psi^3(x) + 2\psi(x) + \psi^3(x) - ax^3.$$

If $a = 0$ then $M[0](x) = 0$ and thus the function $\phi(x) = 0$ is a center manifold (exactly). The reduced equation has the form

$$p' = 0.$$

Therefore the origin is stable, however not asymptotically stable.

If $a \neq 0$, let us find the approximation in the form $\psi(x) = cx^2$. We have

$$M[\psi](x) = 2c^3x^7 + 2cx^2 + c^3x^6 - ax^3.$$

The choice $c = 0$ gives us $M[\psi](x) = -ax^3$, i.e. the center manifold satisfies $\phi(x) = 0 + \mathcal{O}(x^3)$. From this information we can not say anything about the stability of the reduced equation

$$p' = p^2\big(\mathcal{O}(p^3)\big);$$

we have no information about the sign of the right hand side. If we choose $\psi(x) = cx^2$, where $c \neq 0$, the situation is similar because then $M[\psi](x) = \mathcal{O}(x^2)$, i.e. $\phi(x) = cx^2 + \mathcal{O}(x^2)$; once again we can not identify the sign of $\phi(x)$.

Let us use the approximation $\psi(x) = cx^3$. We have

$$M[\psi](x) = 3c^4x^{13} + 2cx^3 + c^3x^9 - 2x^3.$$

The choice $c = 1$ gives us $M[\psi](x) = \mathcal{O}(x^9)$; and so

$$\phi(x) = x^3 + \mathcal{O}(x^9).$$

The reduced equation tells us that

$$p' = p^2\big(p^3 + \mathcal{O}(p^9)\big)^3 = p^2\big(p^9 + \mathcal{O}(p^{15})\big) = p^{11} + \mathcal{O}(p^{17}).$$

And thus the origin is unstable.

Example 3. Examine the stability of the origin of the system

$$\begin{aligned} x' &= x(y - z) \\ y' &= -2y + z + x^2 - z^2 \\ z' &= y - 3z + xyz \end{aligned}$$

Solution. Here we have $n = 1$, $m = 2$; where x is the central variable and (y, z) are the stable variables. The corresponding matrices are $A = 0$ and

$$B = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix},$$

with eigenvalues $(-5 \pm \sqrt{5})/2 < 0$. The center manifold has then the form $y = \phi_1(x)$, $z = \phi_2(x)$; similarly the approximation M has two components:

$$\begin{aligned} M_1[\psi](x) &= \psi_1'(x)x\big(\psi_1(x) - \psi_2(x)\big) + 2\psi_1(x) - \psi_2(x) - x^2 - \psi_2^2(x) \\ M_2[\psi](x) &= \psi_2'(x)x\big(\psi_1(x) - \psi_2(x)\big) - \psi_1(x) + 3\psi_2(x) - x\psi_1(x)\psi_2(x) \end{aligned}$$

Let us try the zero approximation, i.e. $\psi_1 = \psi_2 = 0$. Then $M = \mathcal{O}(x^2)$, and so $\phi_i(x) = \mathcal{O}(x^2)$, for $i = 1, 2$. The reduced equation

$$p' = p\big(\phi_1(p) - \phi_2(p)\big)$$

will not help us. – Let us try the approximation

$$\psi_1(x) = ax^2, \quad \psi_2(x) = bx^2.$$

The first and last summand in M_1, M_2 are at least $\mathcal{O}(x^4)$. Let us try to eliminate the remaining terms, i.e. we want

$$\begin{aligned} 2\psi_1(x) - \psi_2(x) &= x^2 \\ -\psi_1(x) + 3\psi_2(x) &= 0. \end{aligned}$$

The solution is $a = 3/5$, $b = 1/5$. We get an approximation of the c.m. in the form

$$\phi_1(x) = \frac{3}{5}x^2 + \mathcal{O}(x^4)$$
$$\phi_2(x) = \frac{1}{5}x^2 + \mathcal{O}(x^4).$$

The reduced equation gives us

$$p' = p\big(\frac{3}{5}p^2 - \frac{1}{5}p^2 + \mathcal{O}(p^4)\big) = \frac{2}{5}p^3 + \mathcal{O}(p^5).$$

The origin is unstable.

Example 4. Examine the stability of the origin of the system

$$x' = x - \sin y,$$
$$y' = 2\sin x - 2y.$$

Solution. The matrix of the linearized system

$$M = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

has eigenvalues 0 and -1. The corresponding vectors $(1,1)$ and $(1,2)$ determine the central and stable directions respectively. Let us introduce the corresponding variables u, v i.e.

$$x = u + v, \qquad y = u + 2v.$$

The system in the new variables looks as follows

$$u' = 4u + 6v - 2\sin(u + 2v) - 2\sin(u + v),$$
$$v' = -3u - 5v + \sin(u + 2v) + 2\sin(u + v).$$

The center manifold has the form $v = \phi(u)$. The equation for approximation is

$$\begin{aligned} M[\phi](u) = {} & \phi'(u)\big(4u + 6\phi(u) - 2\sin(u + 2\phi(u)) - 2\sin(u + \phi(u))\big) \\ & - \big(-3u - 5\phi(u) + \sin(u + 2\phi(u)) + 2\sin(u + \phi(u))\big). \end{aligned}$$

The choice $\psi(u) = cu^2$ gives us in the best case $c = 0$ the approximation $\phi(u) = \mathcal{O}(u^3)$ which does not help us.

Let's try $\psi(u) = cu^3$. If we consider that

$$\sin(u + au^3) = u + au^3 - \frac{1}{6}(u + au^3)^3 + \mathcal{O}(u^5) = u + (a - \frac{1}{6})u^3 + \mathcal{O}(u^5), \quad (4)$$

we find that the first line in $M[\psi](u)$ is at least $\mathcal{O}(u^5)$. We approximate the second line (without the minus sign) as follows

$$\begin{aligned} &-3u - 5cu^3 + u + 2cu^3 - \frac{1}{6}u^3 + 2(u + cu^3) - \frac{2}{6}u^3 + \mathcal{O}(u^5) \\ &= \big(- 5c + 2c + 2c - \frac{1}{6} - \frac{2}{6})u^3 + \mathcal{O}(u^5). \end{aligned}$$

We eliminate the coefficient of u^3 by choosing $c = -1/2$. Thus the c.m. satisfies $\phi(u) = -u^3/2 + \mathcal{O}(u^5)$. The reduced equation (again using (4)) is in the form

$$\begin{aligned} p' &= 4p + 6\big(- \tfrac{1}{2}p^3 + \mathcal{O}(p^3)\big) - 2\sin(p - p^3 + \mathcal{O}(p^3)) - 2\sin(p - \tfrac{1}{2}p^3 + \mathcal{O}(p^3)) \\ &= 4p - 3p^3 - 2\big(p - p^3 - \frac{1}{6}p^3 + \mathcal{O}(p^3)\big) - 2\big(p - \frac{1}{2}p^3 - \frac{1}{6}p^3 + \mathcal{O}(p^3)\big) \\ &= \frac{2}{3}p^3 + \mathcal{O}(p^3). \end{aligned}$$

The origin is unstable.

Example 5. Examine the behaviour of the system

$$\begin{aligned} x' &= \varepsilon x - x^3 + xy \\ y' &= -y + y^2 - x^2 \end{aligned} \quad (5)$$

on a neighbourhood of the origin for ε close to 0.

Solution. By linearization we get the matrix

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the origin is stable for $\varepsilon < 0$, unstable for $\varepsilon > 0$; for $\varepsilon = 0$ we have a bifurcation in the origin. We now use a trick, we add an equation for ε, i.e.

$$\begin{aligned} \varepsilon' &= 0 \\ x' &= \varepsilon x - x^3 + xy \\ y' &= -y + y^2 - x^2 \end{aligned} \quad (6)$$

This is now a system in the form (1) – with the central variables $X = (\varepsilon, x)$ and the stable variable y; A is the zero matrix 2×2, $b = -1$, $f = (0, \varepsilon x - x^2 + xy)$ and $g = y^2 - x^2$.

Therefore there exists a center manifold $y = \phi(\varepsilon, x)$. The equation for the approximation is

$$\begin{aligned} M[\phi] &= \frac{\partial \phi}{\partial \varepsilon} 0 + \frac{\partial \phi}{\partial x}\big(\varepsilon x - x^3 + x\phi\big) - \big(-\phi + \phi^2 - x^2\big) \\ &= \frac{\partial \phi}{\partial x}\big(\varepsilon x - x^3 + x\phi\big) + \phi - x^2 + \phi^2. \end{aligned}$$

Let us try $\psi = 0$. Then $M[\psi] = -x^2 = \mathcal{O}(|X|^2)$. The reduced equation has the form

$$p' = \varepsilon p - p^3 + p\mathcal{O}(|P|^2) = p\big(\varepsilon - p^2 + \mathcal{O}(|P|^2)\big),$$

where we set $P = (\varepsilon, p)$. The stability of the origin for $\varepsilon = 0$ is still unclear from this.

Let's try a better approximation $\psi = x^2$. Then $M[\psi] = 2x(\varepsilon x) + x^4 = \mathcal{O}(|X|^3)$. The reduced equation has the form

$$p' = p\,\underbrace{\big(\varepsilon - 2p^2 + \mathcal{O}(|P|^3)\big)}_{h(\varepsilon,p)}. \tag{7}$$

Let us remark that $\mathcal{O}(|P|^3)$ here represents a (smooth) function, whose derivative up to and including order *two* are zero in the origin, specially it holds $\mathcal{O}(|P|^3) \leq K(|\varepsilon|^3 + |p|^3)$. From this we see that for $\varepsilon = 0$ the origin is asymptotically stable. More generally, for a function h it holds $h(0,0) = 0$, $\frac{\partial h}{\partial \varepsilon}(0,0) = 1$, $\frac{\partial h}{\partial p}(0,0) = 0$ and $\frac{\partial^2 h}{\partial p^2}(0,0) = -4$. From the implicit function theorem we deduce, that the set $h(\varepsilon, p) = 0$ is a graph of a function $\varepsilon = 2p^2 + \mathcal{O}(p^3)$ on a neighbourhood of the origin; from this we get a bifurcation diagram of the equation (7) – see Figure 1.

We now have a complete idea about the behaviour of solutions of the system (5) on a neighbourhood of the origin depending on ε nearby zero. For $\varepsilon \leq 0$ the origin is asymptotically stable. For $\varepsilon > 0$ the situation is as on the Figure 2. In the direction of the center manifold (in red) the solution gets exponentially close to the center manifold (in gold), which contains, apart from the unstable origin, two stable equilibria (in blue).

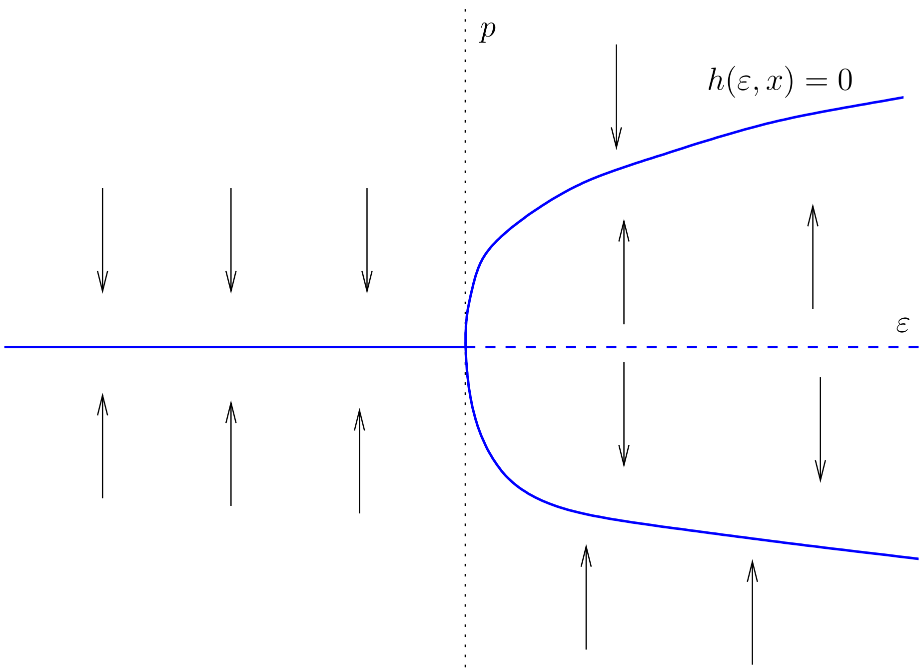

Figure 1: Bifurcation equation (7) : ,,pitchfork“ .

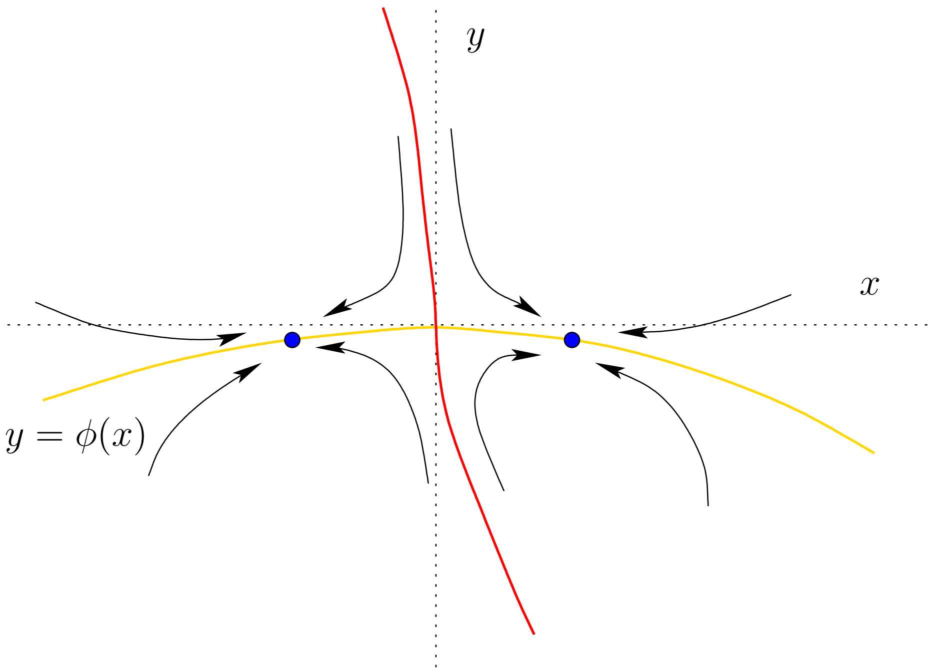

Figure 2: Solution of the equation (5) for $\varepsilon > 0$.

Exercises on the center manifold and its approximation.

1. Show that the system

$$x' = -x^3, \qquad y' = -y$$

has a center manifold $\phi(x) = 0$. Show that

$$\phi(x) = \begin{cases} 0, & x \le 0 \\ \exp(-1/2x^2), & x > 0 \end{cases}$$

is also a center manifold. Find other center manifolds.

2. Show that no center manifold of the system

$$x' = -x^3, \qquad y' = -y + x^2$$

is analytic at 0.

3. Suppose the situation $m = n = 1$, i.e. $A = a \ge 0$, $B = -b < 0$. Let f, g be C^2 and let ϕ be a C^2 function, satisfying $M\phi = 0$ and $\phi(0) = 0$. Show that necessarily $\phi'(0) = 0$ and express $\phi''(0)$ using derivatives of f and g. (Hint: Taylor series.)

4. Let $F(x)$ be continuous on a neighbourhood of 0, $F(0) = 0$ and let $F(x) = ax^n + \mathcal{O}(x^{n+1})$, $x \to 0$, where $a \ne 0$, $n \in \mathbb{N}$. Explore the stability of the point 0 for the equation $x' = F(x)$.

5. Examine the stability of the origin for the system

$$x' = xy, \qquad y' = -y + x^2.$$

6. Examine the stability of the origin for the system

$$x' = xy, \qquad y' = -y - x^2.$$

7. Examine the stability of the origin for the system

$$x' = e^{2xy} - e^{x^3}, \qquad y' = e^{x^2} - e^{2y}.$$

8. Examine the stability of the origin for the system

$$x' = y \sin x + x \sin y, \qquad y' = \ln(1 - y) + \sin x^2.$$

9. Examine the stability of the origin for the system

$$x' = x^2 - 4xy + y^2, \qquad y' = -10y + x^2y^2 + x^5.$$

10. Examine the stability of the origin for the system

$$x' = ax^3 + x^2 y, \qquad y' = -y + y^2 + xy - x^3,$$

where $a \in \mathbb{R}$ is a parameter.

11. Examine the stability of the origin for the system

$$\begin{aligned} x' &= y^2 + z^3, \\ y' &= -y + x^2, \\ z' &= -2z - x^2. \end{aligned}$$

12. Consider the system

$$\begin{aligned} x' &= yv, \\ y' &= yu, \\ u' &= -u + x^2 + 2xy^3, \\ v' &= -v + y^2 + 2x^2 y^2. \end{aligned}$$

Show that there exists a center manifold in the form $u = \phi(x, y)$, $v = \psi(x, y)$. Approximate it by a suitable quadratic function. Examine the stability of the origin of the original system.

13. Consider the system

$$\begin{aligned} x' &= -y + x^2 + yz, \\ y' &= x - y^2, \\ z' &= xy - z. \end{aligned}$$

Check that there exists a c.m. in the form $z = \Phi(x, y)$. Approximate it using a suitable quadratic function. Examine the stability of the reduced equation using the theorem about the stability of the Hopf bifurcation.

14. Examine stability of the origin:

$$\begin{aligned} x' &= -x^3 + 3x^2 yz \\ y' &= -y^3 - 2x^2 yz \\ z' &= -z + 10(x^2 + y^2) \end{aligned}$$

15. Examine stability of the origin:

$$\begin{aligned} x' &= -z^k, \quad k \geq 2 \\ y' &= -y + x^2 \\ z' &= -2z - x^2 \end{aligned}$$

16. Examine stability of the origin:

$$\begin{aligned}
x' &= x(y-z)\\
y' &= -2y + z + z^2 - x^2\\
z' &= y - 3z + xyz
\end{aligned}$$

Solutions.

1) Check that $M[\phi](x) = 0$, i.e. $-x^3\phi'(x) = -\phi(x)$. From that: if $\phi(x)$ is a c.m., then also $c\phi(x)$ is a c.m.

2) By contradiction: let $\phi(x) = \sum_{k=2}^{\infty} a_k x^2$ be a c.m. Then $M[\phi](x) = -x^3\phi'(x) + \phi(x) - x^2 = 0$, which gives us $a_2 = 1$ and generally $a_{k+2} = ka_k$, which is a series with a zero radius of convergence.

3) Let

$$\begin{aligned} \phi(x) &= \beta x + \gamma x^2 \\ f(x,y) &= a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \\ g(x,y) &= b_{11}x^2 + 2b_{12}xy + b_{22}y^2 \end{aligned}$$

be corresponding second order Taylor series. By plugging in (3) we get $\beta = 0$ (coefficient of the x-term) and then $\gamma = b_{22}/(2a+b)$ (coefficient of the x^2-term), and thus

$$\phi''(0) = \frac{\frac{\partial^2 g}{\partial y^2}(0,0)}{2a+b}$$

4) It holds that $x' = ax^n(1 + \mathcal{O}(x))$, therefore x' has the same sign as ax^n in intervals $(-\delta, 0)$, $(0, \delta)$. We have the asymptotic stability for $a < 0$ and n odd; otherwhise 0 is unstable.

5) A suitable approximation is $\psi(x) = cx^2$; by choosing $c = 1$ we get $M[\psi](x) = \mathcal{O}(x^4)$. The reduced equation $p' = p^3 + \mathcal{O}(p^5)$ is unstable.

6) A suitable approximation is $\psi(x) = cx^2$; by choosing $c = -11$ we get $M[\psi](x) = \mathcal{O}(x^4)$. The reduced equation $p' = -p^3 + \mathcal{O}(p^5)$ is asymptotically stable.

7) A suitable approximation is $\psi(x) = cx^2$; by choosing $c = 1/2$ we get $M[\psi](x) = 0$, and thus $\phi(x) = x^2/2$ *is* a c.m. The reduced equation $p' = 0$ is stable, but not asymptotically stable.

8) A suitable approximation is $\psi(x) = cx^2$; by choosing $c = 1$ we get $M[\psi](x) = \mathcal{O}(x^3)$. The reduced equation $p' = 2p^3 + \mathcal{O}(p^4)$ is unstable.

9) The approximation $\psi(x) = 0$ gives us $M[\psi](x) = \mathcal{O}(x^5)$. The reduced equation $p' = p^2 + \mathcal{O}(p^6)$ is unstable.

10) The approximation $\psi(x) = 0$ gives us $M[\psi](x) = \mathcal{O}(x^3)$. The reduced equation $p' = ap^3 + \mathcal{O}(p^5)$ is asymptotically stable for $a < 0$, unstable for $a > 0$.
For $a = 0$ we need a better approximation. It is insufficient to use $\psi(x) = cx^2$; however by choosing $\psi(x) = cx^3$ with $c = -1$ we get $M[\psi](x) = \mathcal{O}(x^4)$. The reduced equation $p' = -p^5 + \mathcal{O}(p^6)$ is asymptotically stable.

11) The center manifold has the form $y = \phi_1(x)$, $z = \phi_2(x)$. The approximation error has the following components

$$M_1[\psi](x) = \psi_1'(x)\big(\psi_1^2(x) + \psi_2^3(x)\big) + \psi_1(x) - x^2,$$
$$M_2[\psi](x) = \psi_2'(x)\big(\psi_1^2(x) + \psi_2^3(x)\big) + 2\psi_2(x) + x^2.$$

By choosing $\psi(x) = 0$ we get $M[\psi](x) = \mathcal{O}(x^2)$, which is insufficient. By choosing $\psi_1(x) = x^2$, $\psi_2(x) = -x^2/2$ we get $M[\psi](x) = \mathcal{O}(x^5)$. From this we get that the reduced equation $p' = p^4 + \mathcal{O}(p^6)$ is unstable.

12) The central directions are (x, y), A is a zero matrix 2×2; the stable directions are (u, v), B is a minus identity matrix. The approximation has two components (in arguments x, y, which we omit) :

$$M_1[\phi, \psi] = \frac{\partial \phi}{\partial x}\psi y + \frac{\partial \phi}{\partial y}\phi y + \phi - x^2 - 2xy^3$$
$$M_2[\phi, \psi] = \frac{\partial \psi}{\partial x}\psi y + \frac{\partial \psi}{\partial y}\phi y + \psi - y^2 - 2x^2y^2$$

It is possible to choose $\phi = x^2$ and $\psi = y^2$, for which $M_1 = M_2 = 0$, i.e. we have the c.m. exactly. The reduced system has the form $p' = q^3$, $q' = p^2 q$ and we see that the solution e.g. in the first quadrant is moving away from the origin in both components. Therefore the origin is not stable.

13) The central directions are $X = (x, y)$, the spectrum of A is $\{\pm i\}$; the stable direction is z, where $B = (-1)$. Further

$$M\Phi = (-y + x^2 + y\Phi)\Phi_x + (x - y^2)\Phi_y + \Phi - xy$$

The approximation in the form $\Psi = a_1 x^2 + 2a_2 xy + a_3 y^2$ resets coefficients of all quadratic terms (i.e. the error is of order $|X|^3$), if and only if $a_1 = -1/5$, $a_2 = 1/10$, $a_3 = 1/5$. The reduced equation (terms up to and including order three) is

$$x' = -y + x^2 - \frac{1}{5}x^2 y + \frac{1}{5}xy^2 + \frac{1}{5}y^3 + \dots$$
$$y' = x - y^2$$

Let us use the Theorem 2 from the chapter ,,Bifurcation“; d is not relevant, because formally $\mu = 0$; further $\omega_0 = 1$ and we compute $a = 1/40$, and thus the origin is not stable.

14) There is a c.m. $z = \phi(x, y)$, for which $\psi = 0$ or $\psi = 10(x^2 + y^2)$ are approximations with second or fourth order error, respectively. Using either of the above approximations, $V = p^2 + q^2$ is a strict Lyapunov function for the reduced equation. Hence asymptotic stability.

15) There is a c.m. (in fact, curve) $y = \phi_1(x)$, $z = \phi_2(x)$. Approximations $\psi_1 = x^2$, $\psi_2 = -x^2/2$ leave the error of order $2k+1 \geq 5$. Reduced equation $p' = -p^{2k}/2^k + \dots$ yields that origin is unstable.

16) There is a c.m. of the form $y = \phi_1(x)$, $z = \phi_2(x)$. Ansatz $\psi_1 = ax^2$, $\psi_2 = bx^2$ leads to $a = -3/5$, $b = -1/5$, with fourth order error. Reduced equation $p' = ap^3 + \dots$ implies that origin is asymptotically stable.