Central limit theorem for functionals of Gibbs particle processes

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Outline

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Particle processes

The outcome space

- $\mathcal{C}^d$ ... the system of all compact subsets in $\mathbb{R}^d$ (equipped with the Hausdorff metric), $\mathcal{C}^{(d)} := \mathcal{C}^d \setminus \{\emptyset\}$
- $\mathbf{N}^d$ ... the space of all configurations (all locally finite, integer valued measures on $\mathcal{C}^{(d)}$)
- $\mathcal{N}^d$ ... the standard σ-algebra on $\mathbf{N}^d$ defined by

$$\mathcal{N}^d = \sigma\left(\left\{\mathbf{x} \in \mathbf{N}^d : \operatorname{card}\{K \in \mathbf{x} : K \in B\} = m\right\}, B \in \mathcal{B}(\mathcal{C}^d) \text{ bounded}, m \in \mathbb{N}\right)$$

Particle process

Definition (Particle process)

Particle process is a point process on $\mathcal{C}^{(d)}$, i.e. a random element

$$\xi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}^d, \mathcal{N}^d)$$

Particle distribution

Reference particle distribution

- Let $\mathbb{Q}$ be a probability measure on $\mathcal{C}^{(d)}$ such that

$$\mathbb{Q}(\{K \in \mathcal{C}^{(d)} : c(K) = 0\}) = 1$$

- Assume $\mathbb{Q}(\{K \in \mathcal{C}^{(d)} : B(K) \subset B(0, R)\}) = 1$ for some $R > 0$

Reference measure

- Let λ be a measure on $\mathcal{C}^{(d)}$ defined by

$$\lambda(B) = \int_{\mathcal{C}^{(d)}} \int_{\mathbb{R}^d} \mathbf{1}_{[K+x \in B]} \, \mathrm{d}x \, \mathbb{Q}(\mathrm{d}K), \; B \in \mathcal{B}(\mathcal{C}^d)$$

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Papangelou conditional intensity

Definition (Papangelou conditional intensity)

Let μ be a point process on $\mathbb{X}$ with distribution P_μ. Let $C^!$ is a measure on $\mathbb{X} \times \mathbf{N}$ that is absolutely continuous with respect to $\sigma \otimes P_\mu$, satisfying for all Borel $B \subset \mathbb{X}$ and $A \in \mathcal{N}$

$$C^![B \times A] = \mathbb{E}\left[\sum_{x \in \mu} \mathbf{1}\{x \in B\}\mathbf{1}\{\mu - \delta_x \in A\}\right].$$

Then the Radon-Nikodým derivative $\lambda^* : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+$ of the measure $C^!$ w.r.t measure $\sigma \otimes P_\mu$ is called the *Papangelou conditional intensity* of the point process μ.

- λ^* is defined to satisfy for every Borel $B \subset \mathbb{X}$ and $A \in \mathcal{N}$

$$C^![B \times A] = \int_B \mathbb{E}\left[\lambda^*(u, \mu)\mathbf{1}\{\mu \in A\}\right] \sigma(\mathrm{d}u).$$

Gibbs particle process

- Gibbs particle process can be defined via Dobrushin-Lanford-Ruelle (DLR) equation or via Georgii-Nguyen-Zessin (GNZ) equation

Definition (Gibbs particle process)

Stationary Gibbs particle process μ with Papangelou conditional intensity λ^*, intensity measure λ and activity $\tau > 0$ is a particle process with distribution $P^{\tau,\beta}$ that satisfies

$$\int_{\mathbf{N}^d} \sum_{K \in \mathbf{x}} f(K, \mathbf{x} \setminus \{K\}) P^{\tau,\beta}(\mathrm{d}\mathbf{x}) = \int_{\mathbf{N}^d} \int_{\mathcal{C}^{(d)}} f(K, \mathbf{x}) \lambda^*(K, \mathbf{x}) \lambda(\mathrm{d}K) P^{\tau,\beta}(\mathrm{d}\mathbf{x})$$

for any measurable function $f : \mathcal{C}^{(d)} \times \mathbf{N}^d \to \mathbb{R}_+$.

- In [Novotná, Beneš], the existence of stationary Gibbs particle process was proved.

Pair potential

Definition (Pair potential)

Pair potential is a measurable, translation invariant function $g : \mathcal{C}^d \to [0, \infty)$, with the property $g(\emptyset) = 0$.

- We deal with the Papangelou intensity λ^* of the form

$$\lambda^*(K, \mathbf{x}) := \tau \exp\left\{-\beta \sum_{L \in \mathbf{x}} g(K \cap L)\right\}, \quad K \in \mathcal{C}^{(d)},\ \mathbf{x} \in \mathbf{N}^d$$

  where $\tau > 0$ is called the activity and $\beta \geq 0$ is called the inverse temperature.

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Innovation

Definition (Innovation)

We define the innovation of a Gibbs particle process μ as a random variable

$$I_\mu(\varphi) = \sum_{K \in \mu} \varphi(K, \mu \setminus \{K\}) - \int_{\mathcal{C}^{(d)}} \varphi(K, \mu)\lambda^*(K, \mu)\lambda(\mathrm{d}K)$$

for any measurable $\varphi : \mathcal{C}^{(d)} \times \mathbf{N}^d \to \mathbb{R}$, for which $|I_{\mathbf{x}}(\varphi)| < \infty$ for μ-a.a. $\mathbf{x} \in \mathbf{N}^d$.

- From the Georgii-Nguyen-Zessin formula: $\mathbb{E}[I_\mu(\varphi)] = 0$
- We are interested in estimates of the Wasserstein distance d_W between an innovation I_μ and a standard Gaussian random variable Z.

Main result

Theorem (Bound on the Wasserstein distance)

Let μ be a stationary Gibbs particle process given by the conditional intensity as above with activity $\tau > 0$, inverse temperature $\beta \geq 0$ and with pair potential g which is bounded from above by some positive constant a. Let $\varphi : \mathcal{C}^{(d)} \to \mathbb{R}$ be a measurable function that does not depend on $\mathbf{x} \in \mathbf{N}^d$ and

$$\varphi \in L^1(\mathcal{C}^{(d)}, \lambda) \cap L^2(\mathcal{C}^{(d)}, \lambda).$$

Then

$$d_W(I_\mu(\varphi), Z) \leq \sqrt{\frac{2}{\pi}} \sqrt{1 - 2\tau(1-\beta b)||\varphi||^2_{L^2(\mathcal{C}^{(d)},\lambda)} + \tau^2||\varphi||^4_{L^2(\mathcal{C}^{(d)},\lambda)}}$$
$$+\tau||\varphi||^3_{L^3(\mathcal{C}^{(d)},\lambda)} + \sqrt{\frac{2}{\pi}}\tau^2||\varphi||^2_{L^1(\mathcal{C}^{(d)},\lambda)}|1 - e^{-\beta a}|$$
$$+2\tau^2||\varphi||^2_{L^2(\mathcal{C}^{(d)},\lambda)}||\varphi||_{L^1(\mathcal{C}^{(d)},\lambda)}|1 - e^{-\beta a}| + \tau^3||\varphi||^3_{L^1(\mathcal{C}^{(d)},\lambda)}|1 - e^{-\beta a}|^2.$$

Next section

The space of all segments

- Let $S \subset \mathcal{C}^{(2)}$ be the space of all segments in $\mathbb{R}^2$ and S_0 be the subsystem of segments centered in the origin.
- Fix a reference probability measure $\mathbb{Q}$ on S_0, which corresponds to $\mathbb{Q}_\phi \otimes \mathbb{Q}_L$, where $\mathbb{Q}_\phi, \mathbb{Q}_L$ are the reference distributions of lengths and directions of segments.
- Recall the assumption

$$\mathbb{Q}(\{K \in \mathcal{C}^{(d)} : B(K) \subset B(0, R)\}) = 1$$

for some $R > 0$, from which the support of $\mathbb{Q}_L$ is $(0, 2R]$.

Gibbs segment process

- Define the pair potential by
$$g(K) := \mathbf{1}\{K \neq \emptyset\}, \quad K \in \mathcal{C}^2$$
Under this setting, g is nonnegative, translation invariant, bounded from above by a and $g(\emptyset) = 0$, ergo satisfies conditions of Theorem 1.

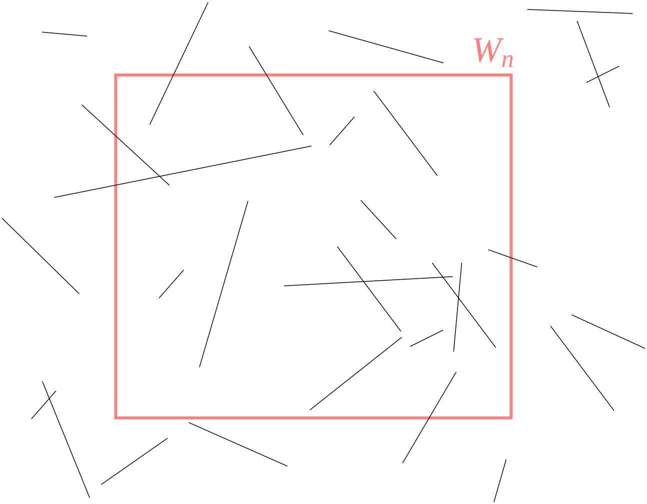
Definition (Gibbs segment process)

We define the Gibbs segment process ξ as a stationary Gibbs particle process in $\mathbb{R}^2$ with the conditional intensity

$$\lambda^*(K, \mathbf{x}) = \tau \exp\left\{-\beta \sum_{L \in \mathbf{x}} \mathbf{1}\{K \cap L \neq \emptyset\}\right\} \quad K \in S, \mathbf{x} \in \mathbf{N}^2.$$

- $\lambda^*(K, \mathbf{x}) = \tau e^{-\beta N_{\mathbf{x}}(K)}$, where $N_{\mathbf{x}}(K)$ denotes the number of intersections of K with segments in $\mathbf{x}$

CLT for a stationary Gibbs segment process



CLT for a stationary Gibbs segment process

Theorem

Consider for each $n \in \mathbb{N}$ a stationary Gibbs planar segment process $\xi^{(n)}$ with the conditional intensity as above with parameters $\tau_n > 0$ and $\beta_n \geq 0$. Suppose that $\beta_n \to 0$ and $0 < c_1 < \tau_n < c_2 < \infty$ for some $c_1, c_2 \in \mathbb{R}$ and that the common reference particle distribution $\mathbb{Q}$ for all $\xi^{(n)}$ has a uniform directional distribution. Let $\{W_n,\ n \in \mathbb{N}\}$ be a convex averaging sequence in $\mathbb{R}^2$ such that $Leb(W_n) = O(\beta_n^{-1})$. For $n \in \mathbb{N}$ and $K \in S$, define

$$\varphi_n(K) = \frac{1}{\sqrt{\tau_n Leb(W_n)}} \cdot \mathbf{1}\{K \cap W_n \neq \emptyset\}, \qquad \psi_n(K) = \frac{l(K)}{\sqrt{\mathbb{E}_L l^2}} \varphi_n(K).$$

where $l(K)$ denotes the length of the segment K, l is a random variable that follows the law of $\mathbb{Q}_L$ and $\mathbb{E}_L$ denotes the expectation with respect to $\mathbb{Q}_L$. Then

$$d_W(I_{\xi^{(n)}}(\varphi_n), Z) \to 0, \qquad d_W(I_{\xi^{(n)}}(\psi_n), Z) \to 0.$$

References

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Thank you for your attention.