Sparse Principal Component Analysis

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Let $\mathbf{X} \in \mathbb{R}^{n \times p}$ be a data matrix and $\widehat{\mathbf{\Sigma}} = \frac{1}{n}\mathbf{X}^\top\mathbf{X}$ be a sample covariance matrix.

For $k = 1, \ldots, p$, let

$$\begin{aligned} \widehat{a}_k \quad = \quad & \underset{a}{\arg\max} \quad a^\top \widehat{\mathbf{\Sigma}} a \\ & \text{s.t.} \quad \|a\|_2^2 \leq 1 \ , \\ & \qquad\;\; a^\top \widehat{a}_i = 0 \ , \ i = 1, \ldots, k-1 \ . \end{aligned}$$

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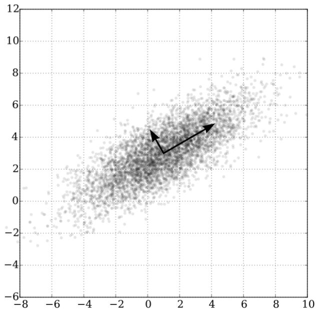
- $\widehat{a}_1, \ldots, \widehat{a}_p$ are *loadings*
- $Y_1 = \mathbf{X}\widehat{a}_1, \ldots, Y_p = \mathbf{X}\widehat{a}_p$ are *principal components* (PCs)
- PCs = uncorrelated standardized linear combinations with the largest possible variance



Zou et al. (2006) showed that loadings can be found as solutions to ridge regression problems

$$\begin{aligned} \widehat{a}_k &= \underset{a}{\arg\max} \quad \|Y_k - \mathbf{X}a\|_2^2 \\ & \quad\text{s.t.} \quad \|a\|_2^2 \leq t \,, \end{aligned}$$

for some t (here completely arbitrary).

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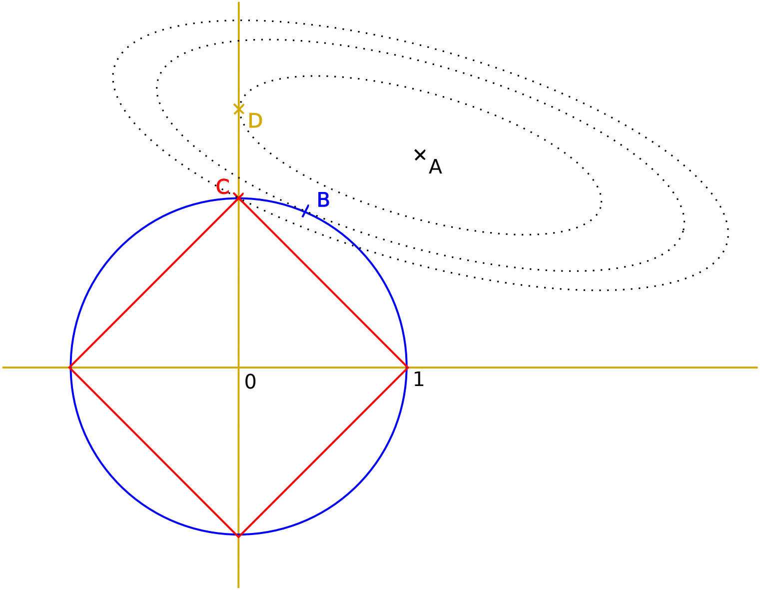
Problems with PCA:

- inconsistency when $p > n$
- poor interpretability – all coefficients non-zero

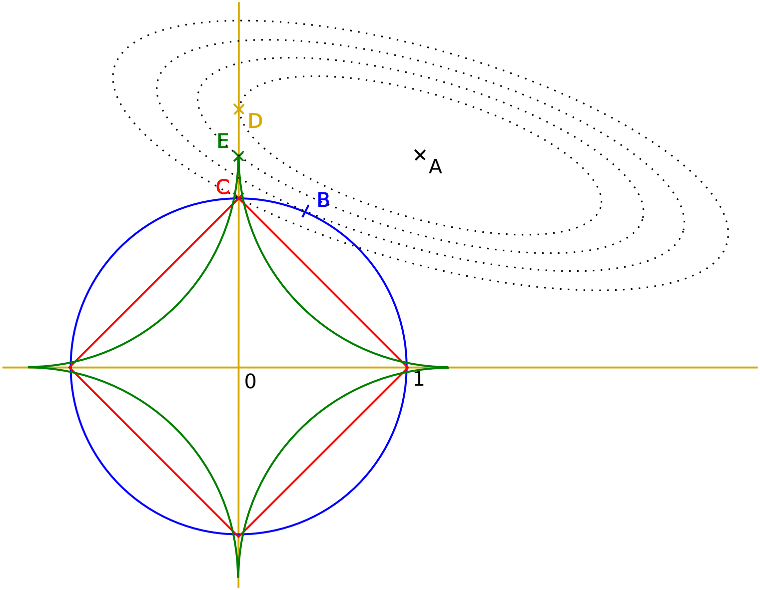
Solution: constraint on cardinality

$$\|a\|_0 \leq s$$

for some s.



$$A: \min_a \|Y_k - \mathbf{X}a\|_2^2$$
$$B: \text{s.t. } \|a\|_2^2 \leq 1$$
$$C: \text{s.t. } \|a\|_1 \leq 1$$
$$D: \text{s.t. } \|a\|_0 \leq 1$$



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$$C: \quad \text{s.t.} \quad \|a\|_1 \leq 1$$
$$D: \quad \text{s.t.} \quad \|a\|_0 \leq 1$$

$$E: \quad \min_a \quad \|Y_k - \mathbf{X}a\|_2^2$$
$$\text{s.t.} \quad \sum_{i=1}^{p} w_i |a_i| \leq 1 \ ,$$

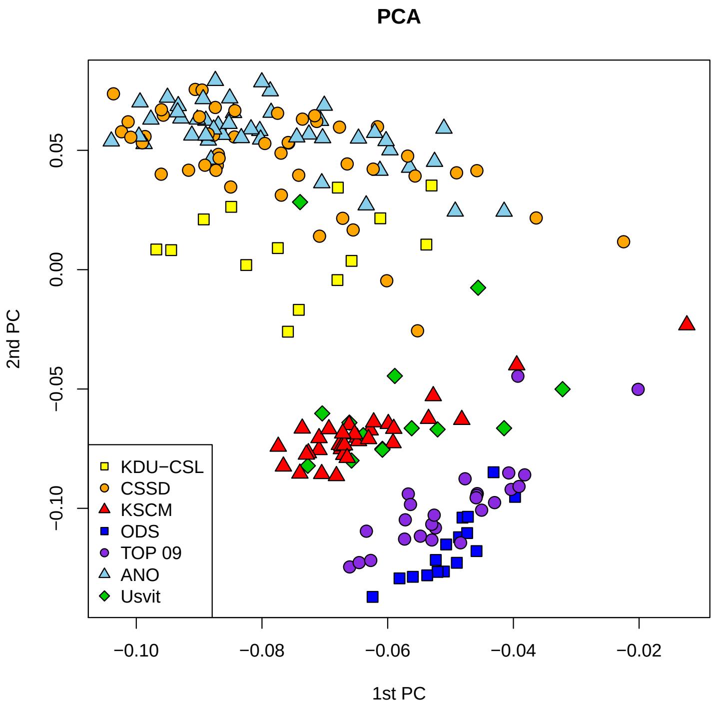
where $w_i = \frac{1}{\tilde{a}_i + \epsilon}$.

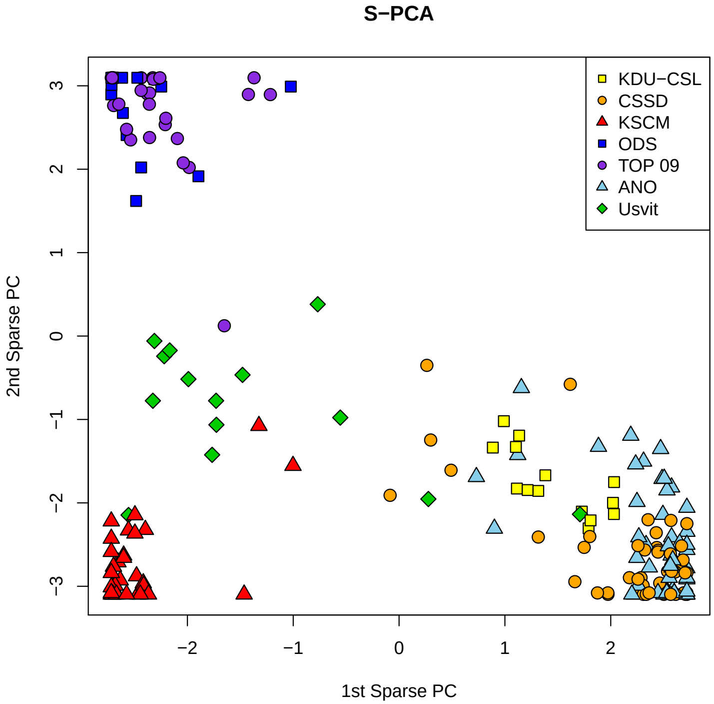
2015 Voting Data

$n = 200$ members of parliament, $p = 1837$ bills voted

- ...
- ...
- Bohuslav Sobotka – vector of votes $x_{Sobotka} \in \{-1, 0, 1\}^p$, and a party affiliation $b \in \{1, \ldots, 5\}$
- ...
- ...

$$x_{Sobotka,j} = \begin{cases} 1, & \text{voted for the } j\text{-th bill} \\ 0, & \text{abstain from voting of the } j\text{-th bill} \\ -1, & \text{voted against the } j\text{-th bill} \end{cases}$$


PCA
2nd PC
1st PC
0.05
0.00
−0.05
−0.10
−0.10
−0.08
−0.06
−0.04
−0.02
KDU−CSL
CSSD
KSCM
ODS
TOP 09
ANO
Usvit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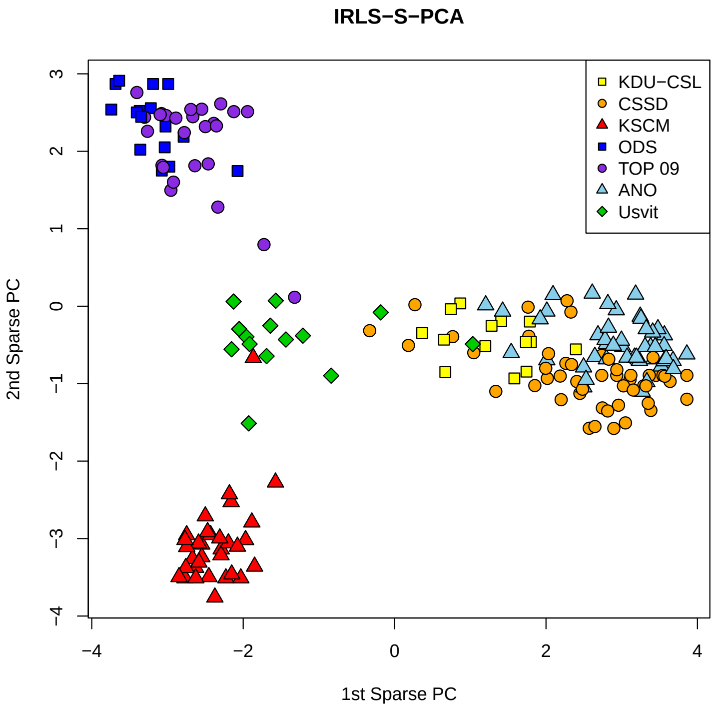
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4

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