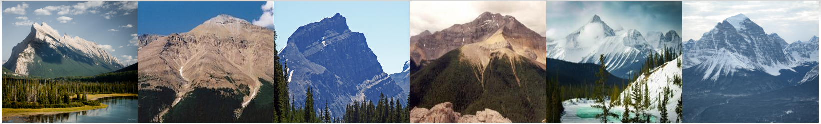
Change-point Estimation and Inference in Nonparametric Regression Using Different Regularization Concepts

Jetřichovice, January 24th, 2014

Matúš Maciak
Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, AB, Canada



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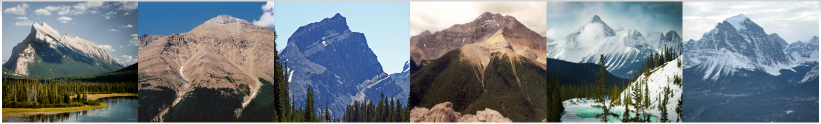

Rundle

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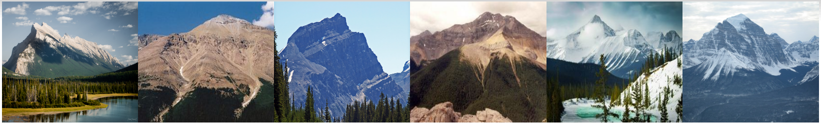

Rundle **O**bservation Pk.

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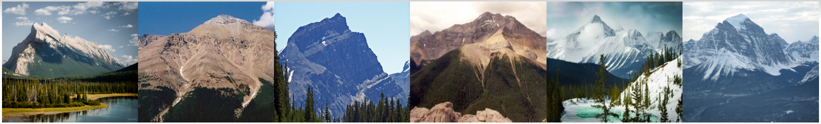

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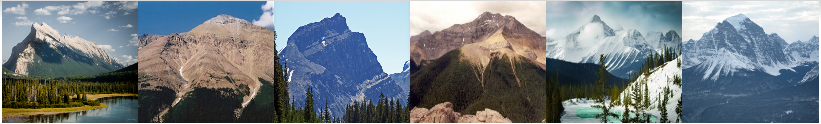

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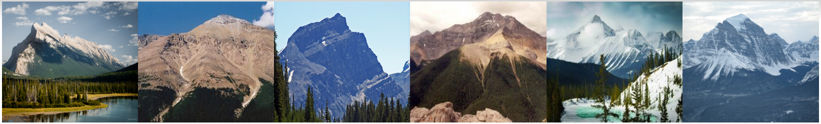

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Rundle


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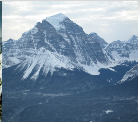
Temple

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University of Alberta, Edmonton, AB, Canada

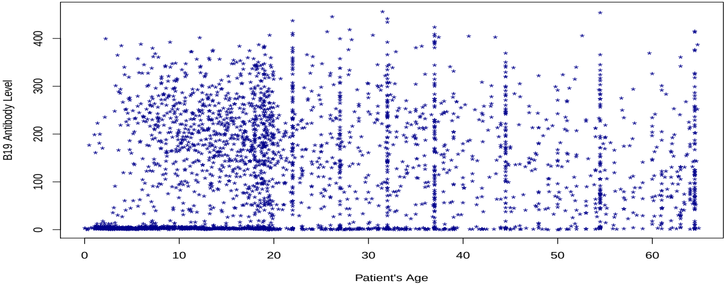
Joint work with Ivan Mizera

Mt. Temple (3.543 meters) July 26, 2013

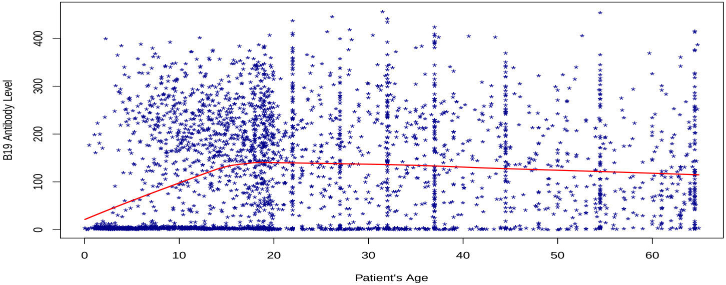
Motivation: Parvovirus B19 Data

- University of Hasselt, Belgium (2008)

Parvovirus B19 Data



Parvovirus B19 Data - LOWESS



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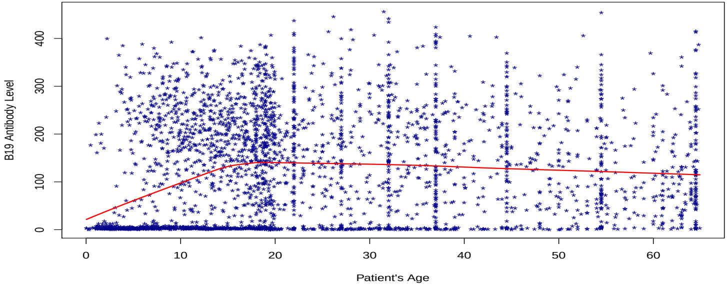
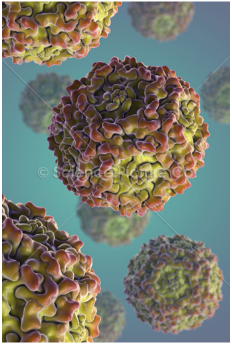


- data analyzed by many authors using various modeling approaches; *(Hens et al. (2010), Maciak (2008), Nardone et al.(2007), etc.)*
- mostly, authors expect some change-points to be present; *(different theoretical and practical limitations)*

Parvovirus B19 Data

- **B19 virus:** mostly known for causing a disease in a pediatric population;
- **Transmission:** respiratory droplets, mostly children at the age of 6 to 10;
- **Infectivity:** individuals after infection generally assumed to be immune;
- **Epidemiology:** increase in the number of cases is seen every three to four years;
- **Data:** over 3000 patients collected in Belgium (November 2001 – March 2003);



Change-points in Regression

- **One-sided estimates** ⇒ **segmented estimation;**
  Antoch et al. (2006); Csörgo and Horváth (1997);
- **Jump detection algorithms** ⇒ **segmented estimation;**
  Horváth and Kokoszka (2002); Qui and Yandell (1998);
- **Permutation tests** ⇒ **segmented estimation;**
  Kim at al. (2009, 2000);
- **Bayesian approach** ⇒ **segmented estimation;**
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- **Total Variation Penalty ⇒ automatic selection using sparsity;**
  Harchaoui and Lévy-Leduc (2010);

The Underlying Model

- random sample $\{(X_i, Y_i);\ i = 1, \ldots, n \in \mathbb{N}\}$, true population $F_{(X,Y)}$;
- the alertdependence structure of Y given X is assumed to take a form

$$Y_i = m(X_i) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

- where function m can be additively decomposed as:

$$m(x) = m_0(x) + \sum_{j=0}^{p-1} s_j(x),$$

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  (smooth function m_0 with some background shock processes $s_0, \ldots, s_{p-1}$)
- $\hookrightarrow$ for identifiability reasons we also assume that

$$\sum_{i=1}^{n} s_j^{(\ell)}(X_i) = 0, \qquad \forall j = 0, \ldots, p-1, \text{ and } \ell = 0, \ldots, j,$$


TIME
AIDS
Debates and
Protests
The
New
Mr.
Dollar
Fed Nominee
Alan Greenspan

Model Estimation Using Splines

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- **Smoothing Splines approach (with change-points):**
  - X_i's observations $\rightarrow$ knots $\{\xi_i;\ i = 1, \ldots, n\}$ $\Rightarrow$ basis functions $\psi_i(x)$;
    $\hookrightarrow$ basis coefficients $\beta_S \in \mathbb{R}^K$, where $m_0(x) = \sum_{i=1}^{K} \beta_S^{(i)} \psi_i(x)$;
  - jump function $s_0(x)$ $\rightarrow$ grid of (hypothetical) jump-locations $\xi_{01}, \ldots, \xi_{0k_0}$;
    $\hookrightarrow$ jump generating basis: zero order truncated basis $\psi_{0j}(x) = (x - \xi_{0,j})_+^0$;
  - ... ...
  - $(p-1)$-order jump function $s_{p-1}(x)$ $\rightarrow$ grid points $\xi_{(p-1)1}, \ldots, \xi_{(p-1)k_{p-1}}$;
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- ideally, we have $k_0 = \cdots = k_{p-1} \equiv k$ and $\xi_{0,j} = \cdots = \xi_{(p-1)j}$ for all $j = 1, \ldots, k$;
- Smoothing spline coefficients $\boldsymbol{\beta}_S$ with a corresponding design matrix $\mathbb{X}_S$ and jump generating (sparse) coefficients $\boldsymbol{\beta}_J$ with a corresponding design matrix $\mathbb{X}_J$;

Minimization formulation

- finite dimensional minimization problem

$$\underset{\beta_S,\, \beta_J}{\textit{Minimize}} \quad \left\| \boldsymbol{Y} - (\mathbb{X}_S \mathbb{X}_J) \begin{pmatrix} \beta_S \\ \beta_J \end{pmatrix} \right\|_2^2 + \lambda_1 \left\| \mathbb{W} \begin{pmatrix} \beta_S \\ \beta_J \end{pmatrix} \right\|_2^2 + \lambda_2 \|\beta_J\|_1$$

- for some $\lambda_1, \lambda_2 > 0$ and $\mathbb{W} = \mathbb{V}^\top \mathbb{V}$, where $\mathbb{V} = (V_{\ell_1 \ell_2})_{\ell_1, \ell_2}$, such that

$$V_{\ell_1 \ell_2} = \int \psi''_{\ell_1}(x) \psi''_{\ell_2}(x) \mathrm{d}x$$

Minimization formulation via LASSO

- for any given $\lambda_1 > 0$ one can apply simple algebra to express the original minimization as

$$\underset{\beta_S,\,\beta_J}{\textit{Minimize}} \quad \left\| \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{0} \end{pmatrix} - \begin{pmatrix} \mathbb{X}_S & \mathbb{X}_J \\ \sqrt{\lambda_1}\mathbb{W}_1 & \sqrt{\lambda_1}\mathbb{W}_2 \end{pmatrix} \begin{pmatrix} \beta_S \\ \beta_J \end{pmatrix} \right\|_2^2 + \lambda_2 \|\beta_J\|_1$$

where $(\mathbb{W}_1, \mathbb{W}_2) = \mathbb{W}$.

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where $(\mathbb{W}_1, \mathbb{W}_2) = \mathbb{W}$.

- defining

$$\mathbb{H} = \begin{pmatrix} \mathbb{X}_S \\ \sqrt{\lambda_1}\mathbb{W}_1 \end{pmatrix} \left[\begin{pmatrix} \mathbb{X}_S \\ \sqrt{\lambda_1}\mathbb{W}_1 \end{pmatrix}^\top \begin{pmatrix} \mathbb{X}_S \\ \sqrt{\lambda_1}\mathbb{W}_1 \end{pmatrix} \right] \begin{pmatrix} \mathbb{X}_S \\ \sqrt{\lambda_1}\mathbb{W}_1 \end{pmatrix}^\top \text{ and}$$

$\mathbb{M} = (\mathbb{I} - \mathbb{H})$, we can express the solution $\mathbb{X}_S\widehat{\beta}_S + \mathbb{X}_J\widehat{\beta}_J$ of the original problem as $\mathbb{H} \begin{pmatrix} \boldsymbol{Y} \\ \mathbf{0} \end{pmatrix} + (\mathbb{I} - \mathbb{H}) \begin{pmatrix} \mathbb{X}_J \\ \sqrt{\lambda_1}\mathbb{W}_2 \end{pmatrix} \widehat{\beta}_J$, where $\widehat{\beta}_J$ solves

$$\underset{\beta_J}{\textit{Minimize}} \quad \left\| \mathbb{M} \begin{pmatrix} \boldsymbol{Y} \\ \mathbf{0} \end{pmatrix} - \mathbb{M} \begin{pmatrix} \mathbb{X}_J \\ \sqrt{\lambda_1}\mathbb{W}_2 \end{pmatrix} \beta_J \right\|_2^2 + \lambda_2 \|\beta_J\|_1$$

Change-point Structure/Hierarchy

- various penalization concepts are possible for in real situations;
- different implementation and interpretations of change-point occurrences;

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- **Mutually Independent Change-points**

- **Simultaneous Change-points**

- **Hierarchical Change-points**

Change-point Structure/Hierarchy

- various penalization concepts are possible for in real situations;
- different implementation and interpretations of change-point occurrences;

- **Mutually Independent Change-points**
  - functions $s_0, \dots, s_{p-1}$ are not (mutually) related;
  - for every $s_j \Rightarrow$ a separate sequence of change-point locations $\xi_{j1}, \dots, \xi_{jk_j}$;

- **Simultaneous Change-points**

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Change-point Structure/Hierarchy

- various penalization concepts are possible for in real situations;
- different implementation and interpretations of change-point occurrences;

- **Mutually Independent Change-points**
  - multiple L_1 penalties - one for each level $(0, 1, \ldots, p-1)$;
  - penalty form: $\lambda_1 \|\beta_J^{(0)}\| + \cdots + \lambda_{p-1} \|\beta_J^{(p-1)}\|$;

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- **Simultaneous Change-points**
  - functions $s_0, \ldots, s_{p-1}$ are all connected through the change-point locations;
  - one sequence of locations $\xi_1, \ldots, \xi_k \Rightarrow$ in each ξ_ℓ every s_j has a "jump";

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- **Simultaneous Change-points**
  - Group LASSO penalty, where each group is defined by the location ξ_ℓ;
  - penalty form: $\lambda \sum_\ell \sqrt{\beta_{0\ell}^2 + \cdots + \beta_{(p-1)\ell}^2}$;

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- **Hierarchical Change-points**
  - lower to higher order discontinuity is considered (change-point hierarchy);
  - if there is a jump in s_j, for some $j = 0, \ldots, p-1 \Rightarrow$ jump in all s_ℓ, for $\ell > j$;

Change-point Structure/Hierarchy

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- **Simultaneous Change-points**
  - Group LASSO penalty, where each group is defined by the location ξ_ℓ;
  - penalty form: $\lambda \sum_\ell \sqrt{\beta_{0\ell}^2 + \cdots + \beta_{(p-1)\ell}^2}$;
- **Hierarchical Change-points**
  - Overlap Group LASSO, where each group is defined by the location ξ_ℓ;
  - penalty form: $\lambda \sum_\ell \inf_g \mathcal{G}(\beta_\ell)$;

Group LASSO vs. Overlap LASSO

$$\mathcal{G}(\boldsymbol{\beta}_l) = \sqrt{\beta_{0l(a)}^2 + \beta_{1l(a)}^2 + \beta_{2l(a)}^2} + \sqrt{\beta_{0l(b)}^2 + \beta_{1l(b)}^2 + \beta_{2l(b)}^2} + \sqrt{\beta_{0l(c)}^2 + \beta_{1l(c)}^2 + \beta_{2l(c)}^2},$$

Group LASSO vs. Overlap LASSO

$$\mathcal{G}(\boldsymbol{\beta}_l) = \sqrt{\beta^2_{0l(a)} + \beta^2_{1l(a)} + \beta^2_{2l(a)}} + \sqrt{\beta^2_{0l(b)} + \beta^2_{1l(b)} + \beta^2_{2l(b)}} + \sqrt{\beta^2_{0l(c)} + \beta^2_{1l(c)} + \beta^2_{2l(c)}},$$

such, that

$$\begin{aligned} \beta_{0l} &= \beta_{0l(a)}, && \text{and} && \beta_{0l(b)} = \beta_{0l(c)} = 0, \\ \beta_{1l} &= \beta_{1l(a)} + \beta_{1l(b)}, && \text{and} && \beta_{1l(c)} = 0, \\ \beta_{2l} &= \beta_{2l(a)} + \beta_{2l(b)} + \beta_{2l(c)}. \end{aligned}$$

LARS Algorithm - geometry

- LARS - Least Angle Regression - Efron et al.(2004)
- straightforward modification to accommodate LASSO approach;

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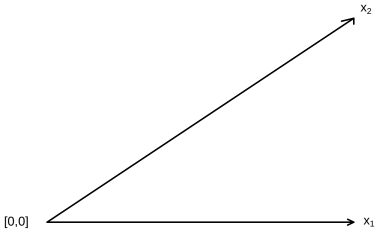

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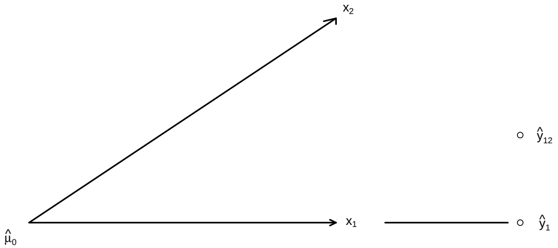

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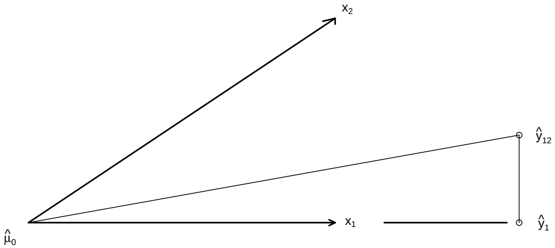

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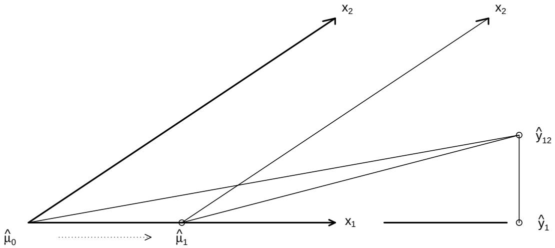

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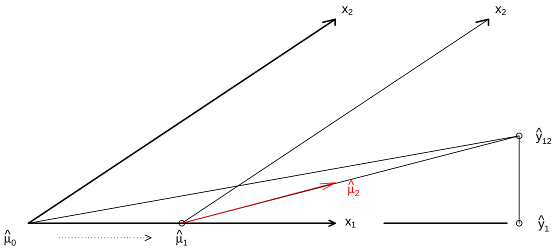
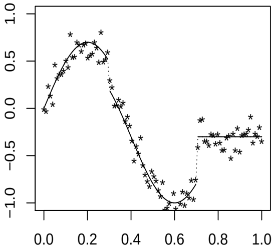


LARS Algorithm - example



- Data: $X_i \sim Unif[0,1]$, for $i = 1, \ldots, 100$;
  $Y_i = m_0(X_i) + \sum_{j=0}^{2} s_j(X_i) + \varepsilon_i$;
- Error: $\varepsilon \sim N(0, 1/400)$;
- Background functions:

$$s_0(x) = 0.49\mathbb{I}(x \geq 0.7) - 0.3\mathbb{I}(x \geq 0.3)$$
$$s_1(x) = -3.9x\mathbb{I}(x \geq 0.7)$$
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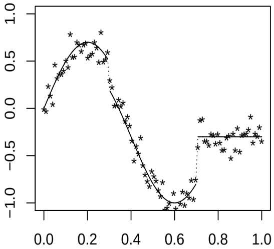

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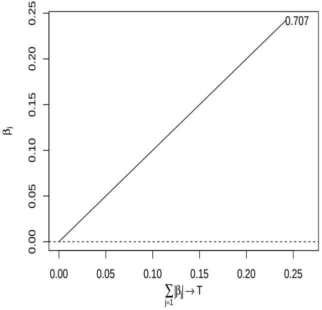
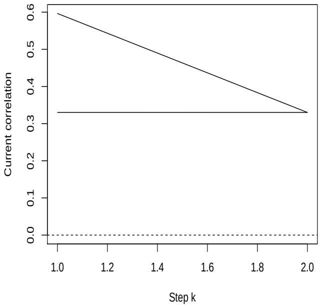
- Mutually Independent Change-points, for $\xi_{0i} = \xi_{1i} = X_i$, $i = 1, \dots, N$;
- Regularization parameters $\lambda_S > 0$ and $\lambda_0, \lambda_1 > 0$;
- LASSO Penalty: $\lambda_0 \|\beta_J^0\| + \lambda_1 \|\beta_J^1\| \longrightarrow$ LARS solution paths;

LARS Algorithm - Solution Paths





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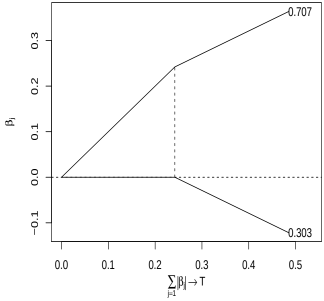

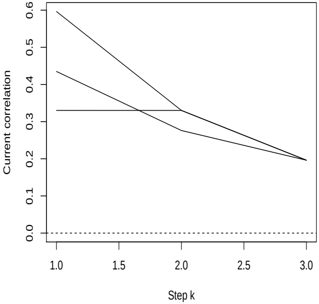

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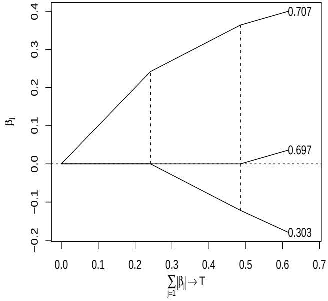

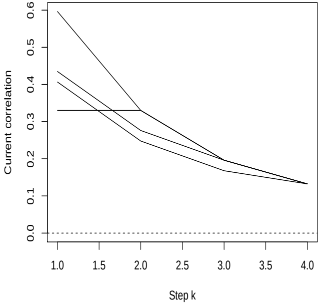

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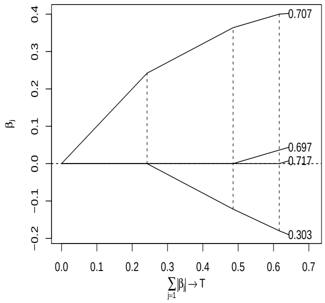

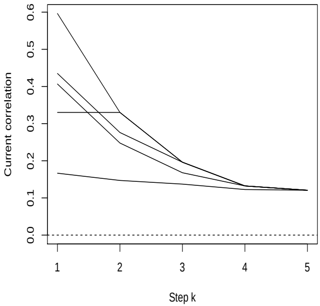

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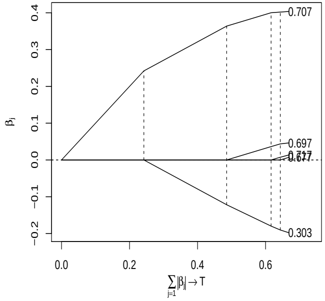

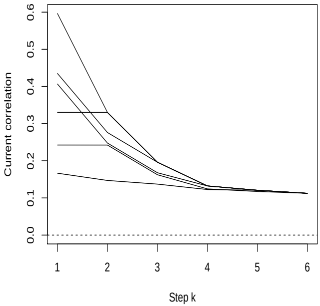

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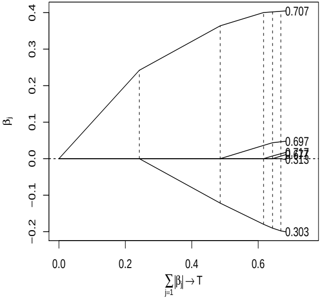

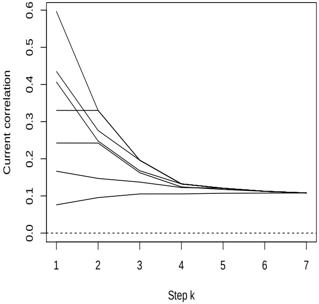

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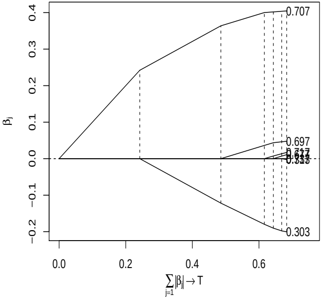

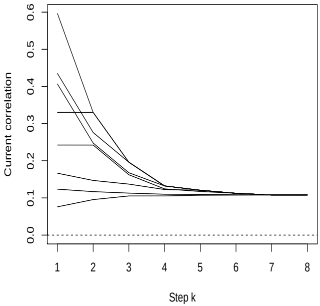

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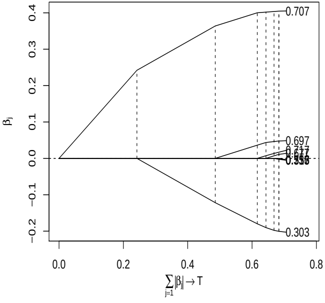

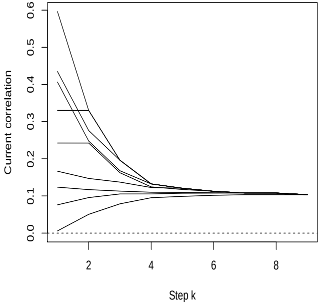

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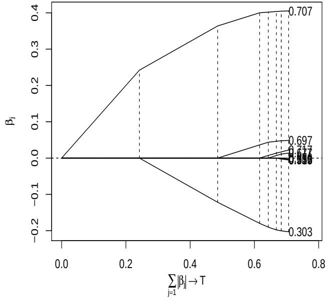

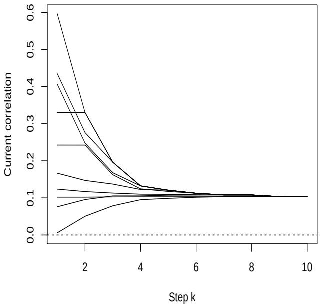

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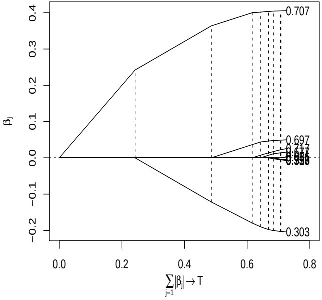

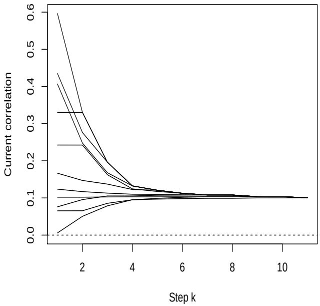

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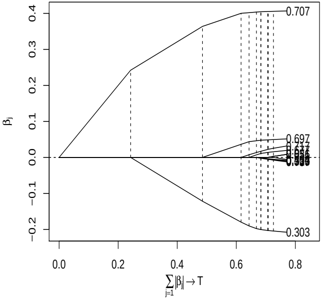

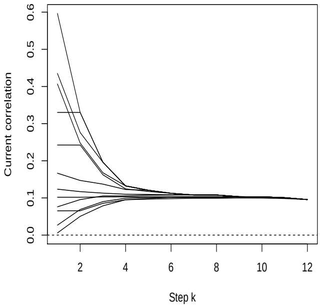

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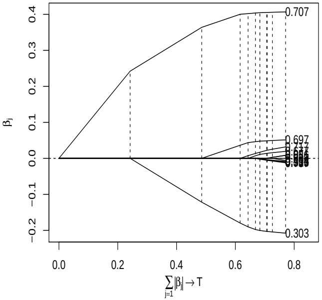

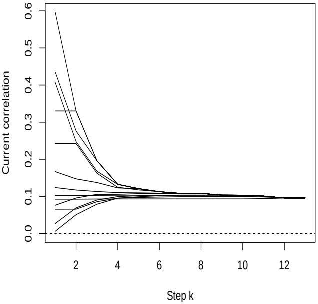

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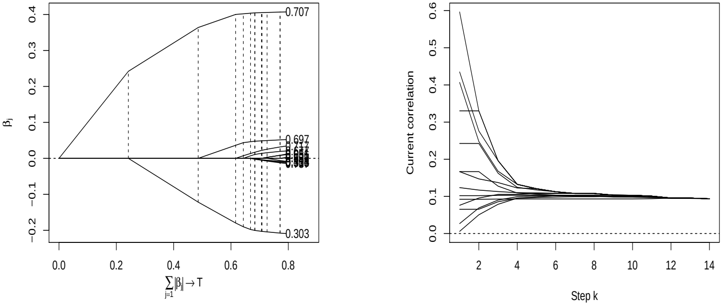

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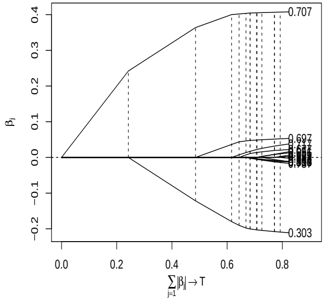

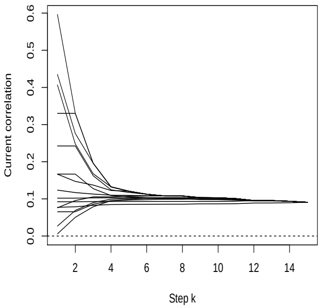

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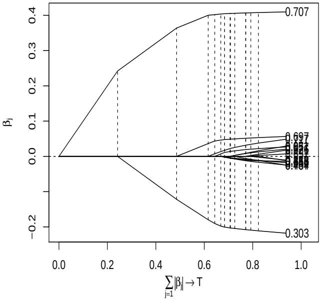

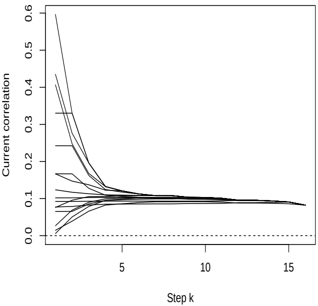

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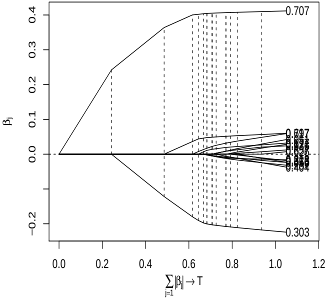

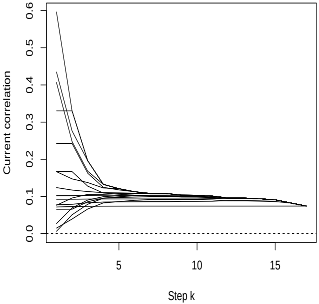

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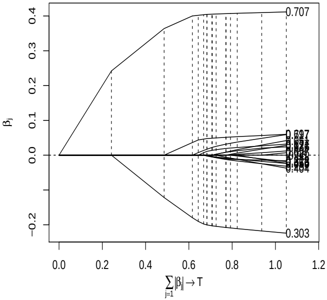

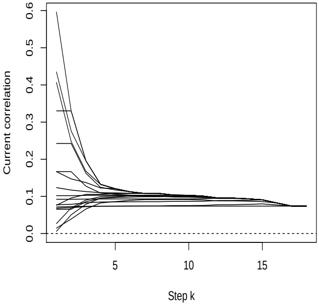

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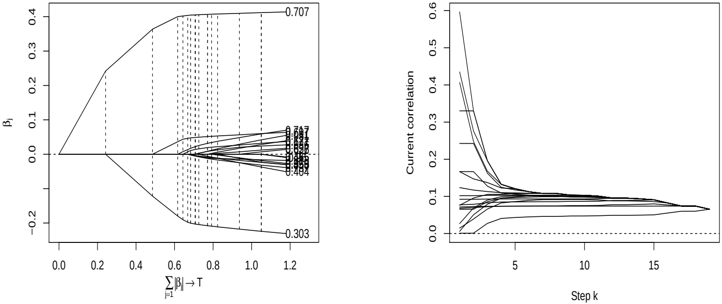

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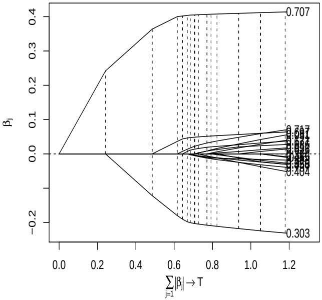

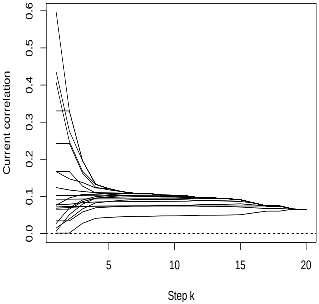

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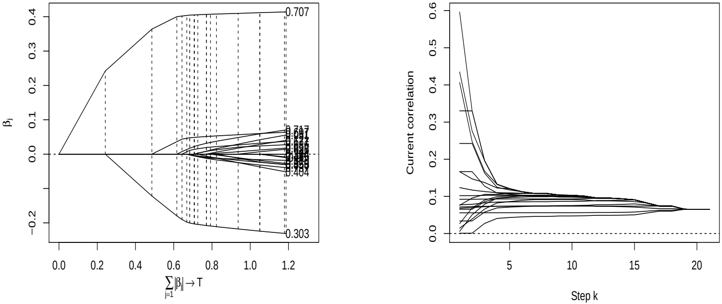

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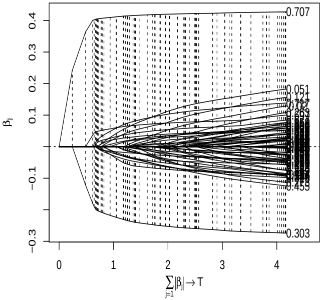

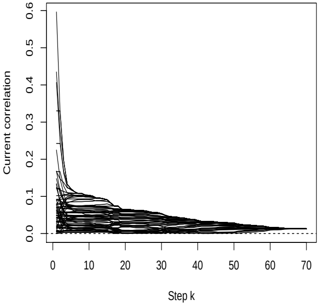

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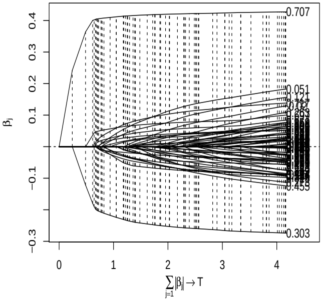
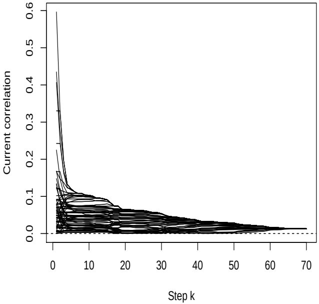
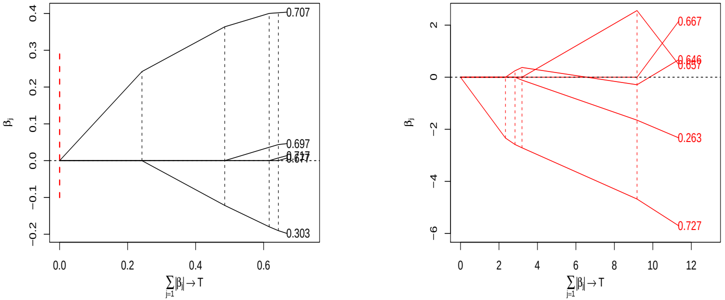




- piece-wise linear solution paths along a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$; (these "knot points" depend on $\boldsymbol{Y}$ and $\mathbb{X}$)
- piece-wise linear decrease in maximum (current) correlation $\mathbb{X}^\top(\boldsymbol{Y} - \widehat{\boldsymbol{\mu}}_k)$;

Multiple Penalties



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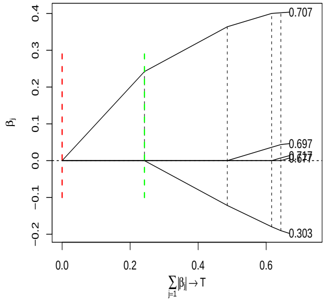

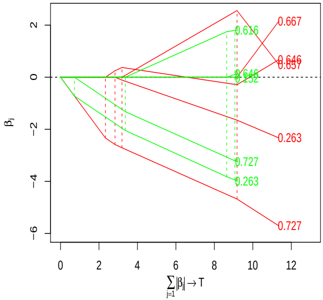

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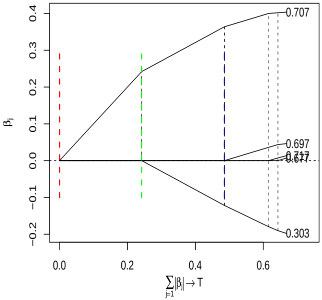

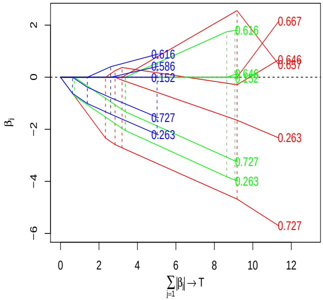

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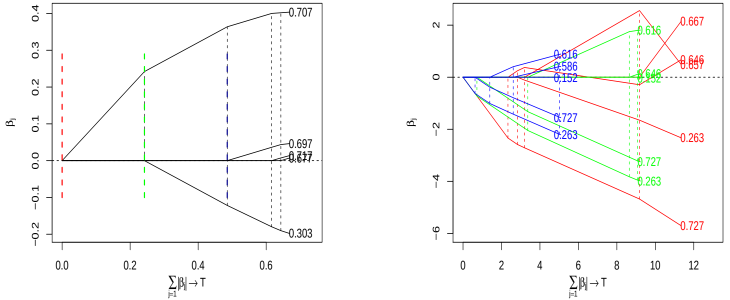

❑ How to choose the final model from the set of plausible ones?

A little bit of inference
on change-points

A little bit of inference

on change-points

- consistency;
- hypothesis tests;
- confidence regions;

Degrees of Freedom

- Degrees of freedom: $df(fit) = \frac{1}{\sigma^2}\sum_{i=1}^{n} Cov\left(\hat{Y}_i, Y_i\right)$;
- linear regression $\Rightarrow$ trace of the hat matrix $\Rightarrow$ number of parameters;
- smoothing splines $\Rightarrow$ trace of $\mathbb{X}\left(\mathbb{X}^\top\mathbb{X} - \sqrt{\lambda_1}\mathbb{W}_1^\top\mathbb{W}_1\right)^{-1}\mathbb{X}^\top$;
- LASSO regression $\Rightarrow$ average number of effective parameters;
  (result generalized by Tibshirani and Taylor (2012) even for $p \geq n$;
- splines with change-points: $\Rightarrow$ hat matrix trace + number of changes;

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  - Mutually Independent Change-points: $df = |\mathcal{A}_0| + \cdots + |\mathcal{A}_{p-1}|$;
  - Simultaneous Change-points: $df = 3 \times |\mathcal{A}|$;
  - Hierarchical Change-points: $df = |\mathcal{A}_0| + \cdots + |\mathcal{A}_{p-1}|$;

Consistency of Estimates

- for now, only consistency with respect to change-points estimates; (considering a model $\widehat{\boldsymbol{\beta}}_J = Argmin\|\boldsymbol{Y} - \mathbb{X}_J\boldsymbol{\beta}_J\|^2 + \lambda\|\boldsymbol{\beta}_J\|_1$)
- restriction on the number of change-points (including their positions); (in general, we assume at most $\mathcal{K} \in \mathbb{N}$ change-points)

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Theorem (Consistency 1)

Under some common assumptions, for all $n \geq 1$ and $C > 2\sqrt{2}$, we have with a probability larger than $1 - n^{1-C^2/8}$, that

$$\left\|\mathbb{X}_J\left(\widehat{\boldsymbol{\beta}}_J(\lambda_n) - \boldsymbol{\beta}_J\right)\right\| \leq (2C\sigma\mathcal{K}\beta_{max})^{1/2} \cdot \left(\frac{\log n}{n}\right)^{1/4},$$

where $\lambda_n = C\sigma\sqrt{\log n/n}$, with an active set of pamameters $\mathcal{A}$.

- idea of the proof: extension of proof in Bickel, Ritov and Tsybakov (2009);

Consistency of Locations

- again, consistency with respect to change-points locations; (considering a model $\widehat{\boldsymbol{\beta}_J} = Argmin\|\boldsymbol{Y} - \mathbb{X}_J\boldsymbol{\beta}_J\|^2 + \lambda\|\boldsymbol{\beta}_J\|_1$)
- two change-point locations are not too much close to each other; (in general, we need enough data points to estimate each change-point)

Theorem (Consistency 2)

It can be shown that

$$\mathbb{P}\left(max_{1\leq k\leq|\widehat{\mathcal{A}}(\lambda_J)|}|\widehat{t}_k - t_k^\star| \leq n\delta_n\right) \overset{n\to\infty}{\longrightarrow} 1,$$

for some nonincreasing, positive sequence $\{\delta_n\}_{n\geq 1}$ *tending to zero, such that* $n\delta_n$

- generalization of the result of Harchaoui and Lévy-Leduc (2010)

Significance Test for LASSO

- classical theory based on RSS drop between two models not applicable;
  $\hookrightarrow$ test statistics: $R_j = (RSS_M - RSS_{M \cup \{j\}})/\sigma^2 \rightarrow \chi^2$ distribution
- in situations where $p \geq n$ the sets M and $M \cup \{j\}$ are not fixed any more;
  $\hookrightarrow$ using classical approach is way too far liberal (large type I. error)
- alternative approach must account for adaptivity of the LASSO procedure;
  $\hookrightarrow$ adaptiveness vs. shrinkage
- covariance test statistic proposed by Lockhart et at. (2013);
  $\hookrightarrow$ test statistics: $T_k = \left(\langle \boldsymbol{Y}, \mathbb{X}\widehat{\beta}(\lambda_{k+1}) \rangle - \langle \boldsymbol{Y}, \mathbb{X}_{\mathcal{A}}\widetilde{\beta}_{\mathcal{A}}(\lambda_{k+1}) \rangle \right) / \sigma^2$
- under the null hypothesis ($supp(\beta^\star) \subseteq \mathcal{A}$) it holds that:
  $\hookrightarrow$ test statistic $T_k \longrightarrow Exp(1)$ in distribution;

Confidence Regions

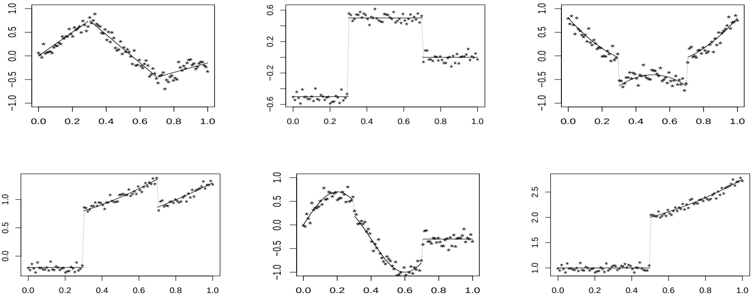
Point-wise Confidence Bands

- for the vector of parameters $(\beta_S^\top, \beta_J^\top)$ we have a pseudo design matrix;
- we can define a sandwich estimate for the covariance matrix;
- if variance σ^2 is unknown $\Rightarrow$ need for an estimate $\widehat{\sigma_n^2}$;

Uniform Confidence Bands

- the idea is to obtain a band $B_n(x)$ for m_0 ($s_0, \ldots, s_{p-1}$ resp.), such that $\mathbb{P}(f(x) \in B_{n,f}(x)) = 1 - \alpha$, for $f \in \{m_0, s_0, \ldots, s_{p-1}\}$;
- idea of the band construction: Hotelling (1939);
  (also Krivobokova et al. (2013) and Koenker (2011))
- however, requires continuity at least $\Rightarrow$ not applicable for s_0 yet;

Some Simulation Results



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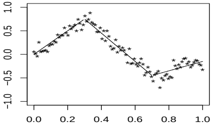

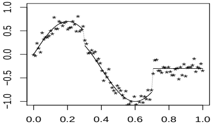

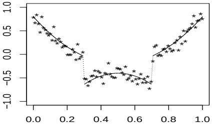

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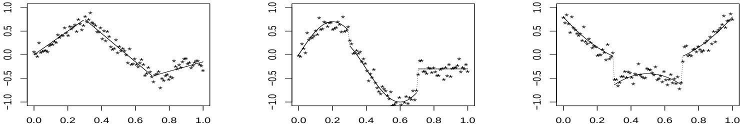

Mutually independent change-points

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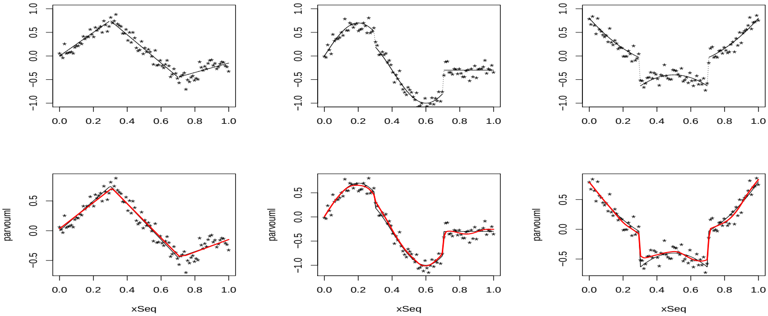

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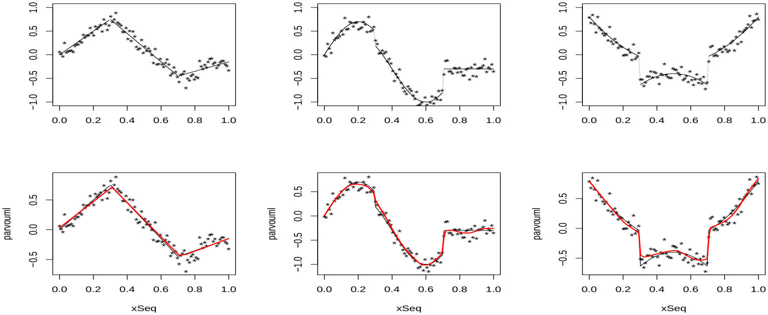
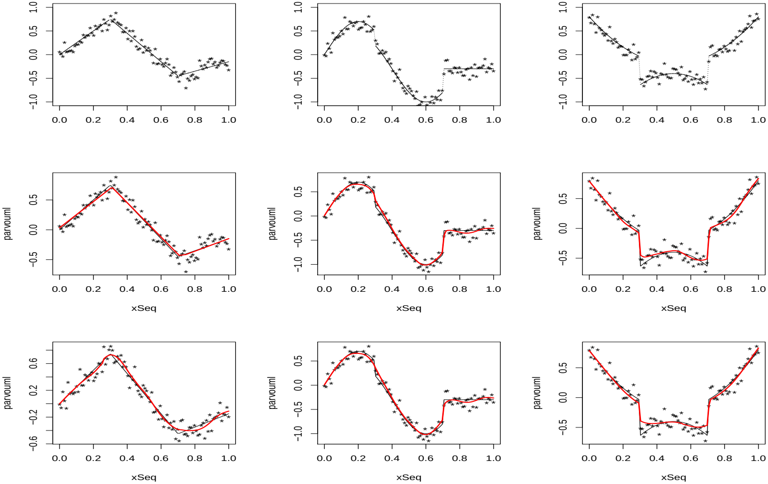


Simultaneous change-points

Some Simulation Results



Independent Change-points

	Independent Change-points	$\sigma^2 = 0$	$\sigma^2 = 0.1$	$\sigma^2 = 0.2$	$\sigma^2 = 0.5$	$\sigma^2 = 1$
$\lambda_G = 0.1$	$\xi_1^{(0)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	3.8 | 1.8	49.4 | 25.8	69.6 | 38.5	94.5 | 49.3
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	69.2 | 31.8	85.8 | 58.5.0	100 | 69.2	100 | 68.1
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	100 | 12.2	100 | 31.5	100 | 37.6	100 | 52.9
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7,\ \xi_1^{(1)} = 0.5$	0.0 | 0.0	100 | 24.5	100 | 37.2	100 | 41.2	100 | 59.5
$\lambda_G = 0.01$	$\xi_1^{(0)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	2.6 | 1.0	38.5 | 20.18	65.8 | 35.4	82.1 | 49.1
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	71.0 | 33.3	87.2 | 60.0	100 | 63.6	100 | 68.5
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	100 | 30.4	100 | 55.4	100 | 60.2	100 | 66.7
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7,\ \xi_1^{(1)} = 0.5$	0.0 | 0.0	100 | 38.7	100 | 58.4	100 | 64.7	100 | 69.2
$\lambda_G = 0.001$	$\xi_1^{(0)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	3.3 | 1.4	52.3 | 27.0	77.9 | 43.7	97.6 | 57.0
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	82.2 | 56.8	92.2 | 70.8	100 | 75.3	100 | 77.3
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	100 | 58.7	100 | 73.7	100 | 75.4	100 | 78.2
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7,\ \xi_1^{(1)} = 0.5$	0.0 | 0.0	100 | 61.2	100 | 77.1	100 | 80.2	100 | 79.1
$\lambda_G = 0.0001$	$\xi_1^{(0)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	2.6 | 0.9	62.5 | 35.2	87.4 | 52.9	98.7 | 59.3
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7$	0.0 | 0.0	98.7 | 70.2	99.9 | 71.7	100 | 76.6	100 | 76.6
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	100 | 75.2	99.9 | 74.6	100 | 77.1	100 | 76.2
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3,\ \xi_2^{(0)} = 0.7,\ \xi_1^{(1)} = 0.5$	0.0 | 0.0	99.9 | 71.0	99.9 | 69.9	100 | 78.2	100 | 79.1

Mutually Related Change-points

	Mutually Related Change-points	$\sigma^2 = 0$	$\sigma^2 = 0.1$	$\sigma^2 = 0.2$	$\sigma^2 = 0.5$	$\sigma^2 = 1$
$\lambda_G = 0.1$	$\xi_1^{(0)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	4.1 | 2.8	51.4 | 27.7	70.6 | 40.5	90.0 | 50.3
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7, \xi_1^{(1)} = 0.5$	100	100	100	100	100
$\lambda_G = 0.01$	$\xi_1^{(0)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	3.8 | 3.1	49.1 | 29.3	72.1 | 40.5	93.7 | 51.9
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7, \xi_1^{(1)} = 0.5$	100	100	100	100	100
$\lambda_G = 0.001$	$\xi_1^{(0)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	4.0 | 2.2	55.5 | 29.2	74.2 | 45.5	97.2 | 54.0
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7, \xi_1^{(1)} = 0.5$	100	100	100	100	100
$\lambda_G = 0.0001$	$\xi_1^{(0)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7$	100	100	100	100	100
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = \xi_2^{(1)} = 0.7$	0.0 | 0.0	3.0 | 1.2	63.1 | 37.3	85.2 | 55.4	98.3 | 59.9
	$\xi_1^{(0)} = \xi_1^{(1)} = 0.3, \xi_2^{(0)} = 0.7, \xi_1^{(1)} = 0.5$	100	100	100	100	100

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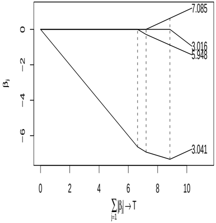
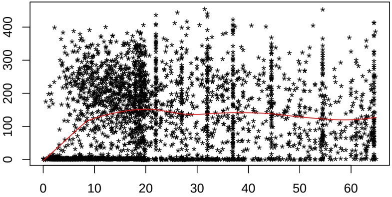
- ❑ intercept corrected smoothing B-spline basis + change-point basis;
  $\Rightarrow$ smoothness degree $p = 3$, change-points up to the order $p - 1 = 2$;
- ❑ mutually independent change-points assumed;
  $\Rightarrow$ four smoothing parameters $\lambda_S, \lambda_0, \lambda_1, \lambda_2 > 0$;





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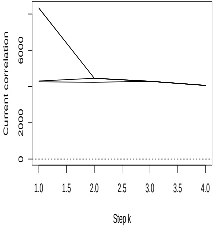
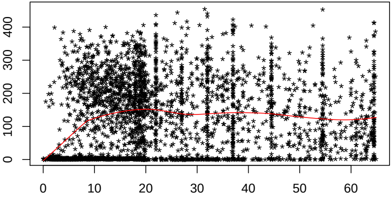
- intercept corrected smoothing B-spline basis + change-point basis;
  ⇒ smoothness degree $p = 3$, change-points up to the order $p - 1 = 2$;
- mutually independent change-points assumed;
  ⇒ four smoothing parameters $\lambda_S, \lambda_0, \lambda_1, \lambda_2 > 0$;





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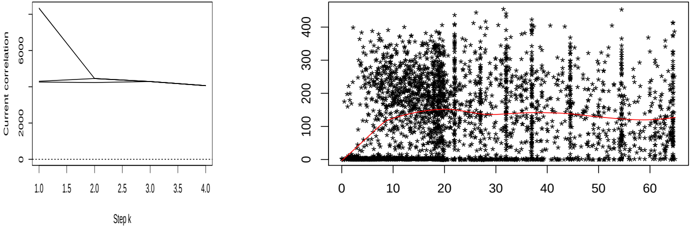
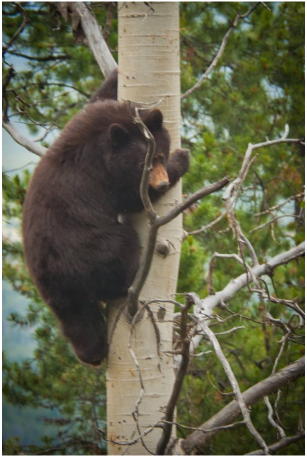
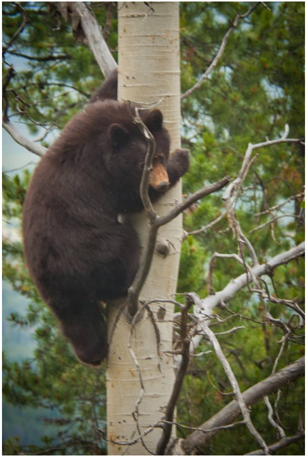
- intercept corrected smoothing B-spline basis + change-point basis;
  ⇒ smoothness degree $p = 3$, change-points up to the order $p - 1 = 2$;
- mutually independent change-points assumed;
  ⇒ four smoothing parameters $\lambda_S, \lambda_0, \lambda_1, \lambda_2 > 0$;

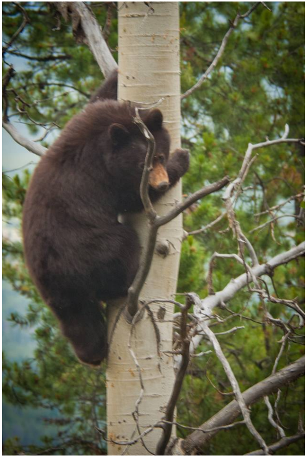


- one change-point in location (roughly at the age of 20);
- in addition, also a change-point in direction revealed (age 8 – 9);
  (even significant - p-value below 0.05)





Thank you...



Thank you...

...any questions?

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