# INTERPOLATION AND THE BELTRAMI EQUATION

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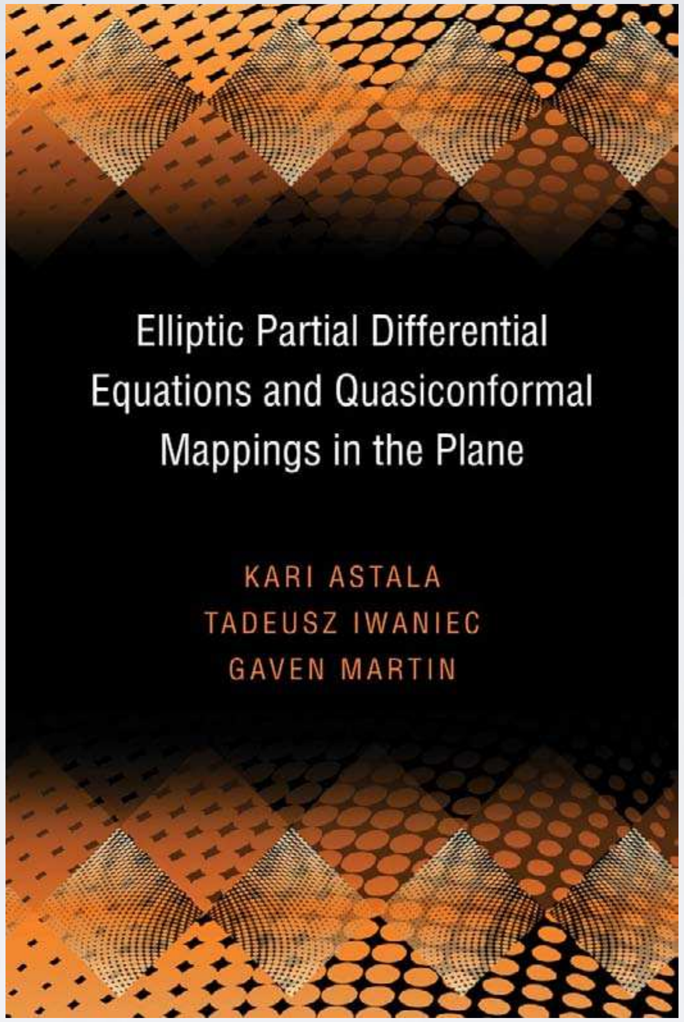
Prague, September 2011

# Quasiconformal mappings in the plane




Beltrami equation

Beurling transform

# Beltrami equation

$$f_{\bar z} = \mu(z)\, f_z, \quad |\mu(z)| \leqslant k\, \chi_D(z), \quad 0 \leqslant k < 1$$

$f(z) = z + \mathcal{O}(1/z)$ principal *quasiconformal mapping*

$f \in W^{1,2}_{loc}(\mathbb{C})$ homeomorphism

$$K = \frac{1+k}{1-k}$$

$J(z,f) > 0$ a.e.

**RIEMANN'S MAPPING THEOREM FOR VARIABLE METRICS***

existence and uniqueness

Morrey

regularity

Bojarski, Astala, etc.

analytic dependence

Ahlfors-Bers

# Beurling transform

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2} dm(\zeta)$$

$S(f_{\bar{z}}) = f_z, \quad f \in W^{1,2}(\mathbb{C})$ $L^2$-isometry

$$f_{\bar{z}} = (Id - \mu S)^{-1}(\mu) = \mu + \mu S\mu + \mu S(\mu S\mu) + \cdots$$

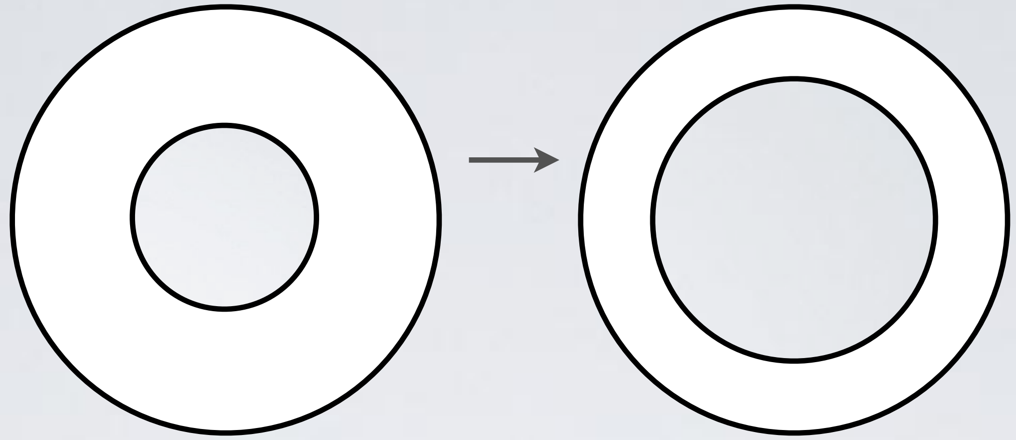
Calderón-Zygmund: $S : L^p \to L^p$

Riesz-Thorin: some $p_0 > 2, \quad \|S\|_{p_0} = \frac{1}{k}$

Bojarski (1957): $f \in W^{1,p}_{loc}(\mathbb{C}), \quad 2 \leq p < p_0(K)$

# Critical Sobolev exponent

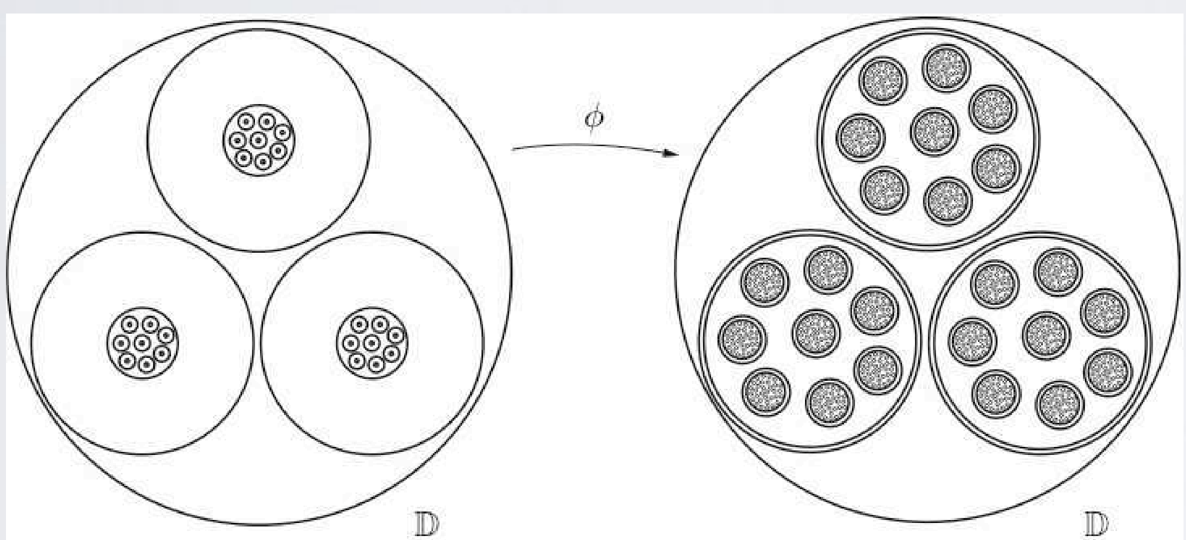
$$z \mapsto \frac{z}{|z|}\,|z|^{\frac{1}{K}}$$

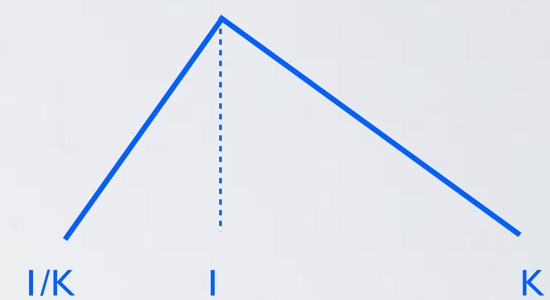


$$p_0(K) = \frac{2K}{K-1}$$

Sobolev embedding →

$\frac{1}{K}$ Hölder exponent (Mori)




$$\dim_H\{z \in \mathbb{C} : \alpha(z) = \alpha\} \leq 1 + \alpha - \frac{|1-\alpha|}{k}$$

Astala (1994): "this is the worst case scenario"

Conjecture (Iwaniec): $\|S\|_{L^p(\mathbb{C})} = p - 1, \quad p \geqslant 2$

# Holomorphic motions

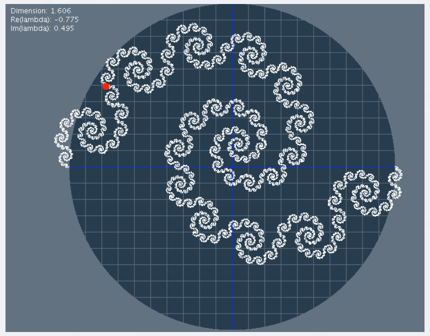
$$\Phi\colon \mathbb{D} \times E \to \mathbb{C}, \quad E \subset \mathbb{C}$$

- $z \mapsto \Phi(\lambda, z) = \Phi_\lambda(z)$ is injective for all $\lambda \in \mathbb{D}$,
- $\lambda \mapsto \Phi(\lambda, z)$ is holomorphic for all $z \in E$,
- $\Phi(0, z) \equiv z$.

Mañé-Sad-Sullivan, Slodkowski's λ-lemma:

"holomorphic motions = quasiconformal maps"

$\{\Phi_\lambda(z)\}$ (extends to) an analytic family of qc maps




java animation by Aleksi Vähäkangas

# Complex interpolation

λ-lemma:

quasiconformal maps = "complex interpolation class"

| | | |
|---|---|---|
| homeomorphisms | | $L^2$ |
| quasiconformal maps | complex interpolation | $L^p$ |
| diffeomorphisms | | $L^\infty$ |

# Interpolation theme

| | | | |
|---|---|---|---|
| analytic dependence | $L^p$ functions | Beltrami equation/ holomorphic motions | |
| interpolate | Beurling transform | dimension/ pressure | $p$-norm of gradient |
| end-point estimates | $L^2$-isometry | dim $\leq 2$ | Jacobian null-Lagrangian |
| non-vanishing | - | injectivity | Jacobian positive a.e. |
| application | gain in regularity *Bojarski* | sharp exponent, area/dimension distortion *Astala* | sharp weight, quasiconvexity *Astala-Iwaniec-Prause-Saksman* |

# Sharp weighted integrability

Astala-Iwaniec-Prause-Saksman

$$f_{\bar z} = \mu(z)\, f_z, \quad |\mu(z)| \leqslant k\, \chi_\Omega(z), \quad 0 \leqslant k < 1, \quad f(z) = z + \mathcal{O}(1/z)$$

$$2 \leq p \leq 1 + 1/k$$

**Theorem:**
*(JAMS, to appear)*

$$\frac{1}{|\Omega|} \int_\Omega \left( 1 - p\, \frac{|\mu(z)|}{1 + |\mu(z)|} \right) |Df(z)|^p \leqslant 1$$

- sharp weight, sharp constants
- "localized integrability" at the borderline
- partial quasiconvexity

rank-one convexity vs quasiconvexity

# Rank-one convexity vs quasiconvexity

*local* **Morrey** *global*

$$\mathbf{E}: \mathbb{R}^{n\times n} \to \mathbb{R}$$

$$\operatorname{rank} X = 1$$
$$t \mapsto \mathbf{E}(A + tX) \text{ convex} \qquad \Leftarrow \qquad \int_\Omega \mathbf{E}(Df) \geqslant \int_\Omega \mathbf{E}(A) = \mathbf{E}(A)|\Omega|$$

$$f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$$

*(ellipticity of Euler-Lagrange)* *(lower semicontinuity)*

$n \geqslant 3$ Šverák $\not\Rightarrow$

$n = 2$ ? Faraco-Székelyhidi: “localization”

Burkholder: $B_p(Df) = B_p(f_z, f_{\bar z})$ rank-one concave

$$B_p(A) = \left( \frac{p}{2} \det A + \left(1 - \frac{p}{2}\right) |A|^2 \right) \cdot |A|^{p-2}$$

# Martingale inequality

## Burkholder

$X_n \prec Y_n$ subordinated martingales

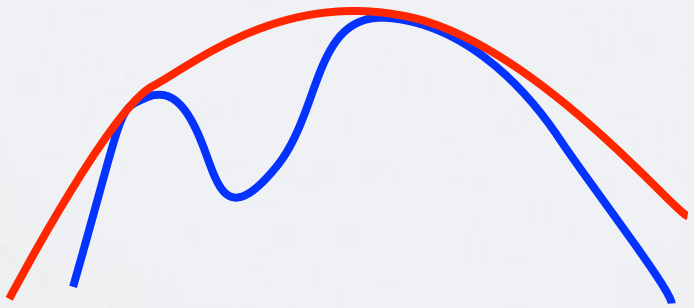
$$|X_n - X_{n-1}| \le |Y_n - Y_{n-1}| \ \text{a.s.}$$

$$\Rightarrow \quad \|X_n\|_p \leqslant (p-1)\,\|Y_n\|_p.$$

$$B_p(z,w) = \left(|z| - (p-1)\,|w|\right)\cdot\left(|z| + |w|\right)^{p-1}$$

$$|z|^p - (p-1)^p\,|w|^p \ \leqslant\ c_p\, B_p(z,w)$$

$$\mathbb{E}\, B_p(X_n, Y_n) \leqslant \mathbb{E}\, B_p(X_{n-1}, Y_{n-1}) \leqslant \ldots \leqslant 0$$



# Quasiconvexity result

$$B_p(z,w) = \left(|z| - (p-1)\,|w|\right)\cdot\left(|z| + |w|\right)^{p-1} \qquad p \geqslant 2$$

$$B_p(Df) = B_p(f_z, f_{\bar z})$$

**Theorem:** $f(z) \in z + C_0^\infty(\Omega), \quad B_p(Df) \geqslant 0, \; z \in \Omega$

**(equiv.)**

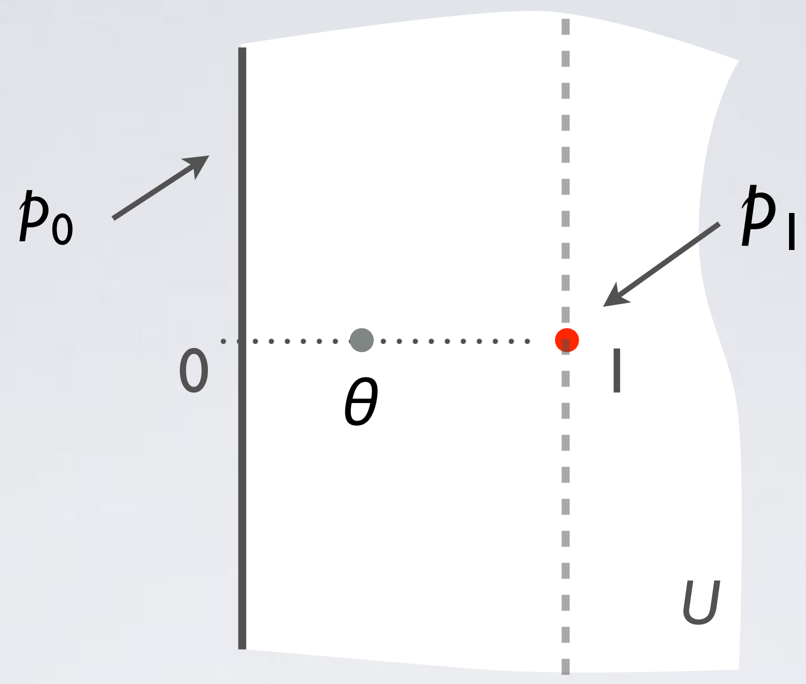
$$\int_\Omega B_p\,(Df) \leqslant \int_\Omega B_p\,(Id) = |\Omega|$$

Burkholder's martingale inequality

$$\mathbb{E}\,B_p(X_n, Y_n) \leqslant 0 \quad \text{for} \quad X_n \prec Y_n$$

full quasiconvexity → $\|S\|_{L^p(\mathbb{C})} = p-1$

# Interpolation lemma




$$0 < p_0, p_1 \leqslant \infty, \quad \theta \in (0,1)$$

$\phi_\lambda(z)$ analytic family, $\lambda \in U = \{\mathrm{Re}\,\lambda > 0\}$

non-vanishing $\phi_\lambda(z) \neq 0$

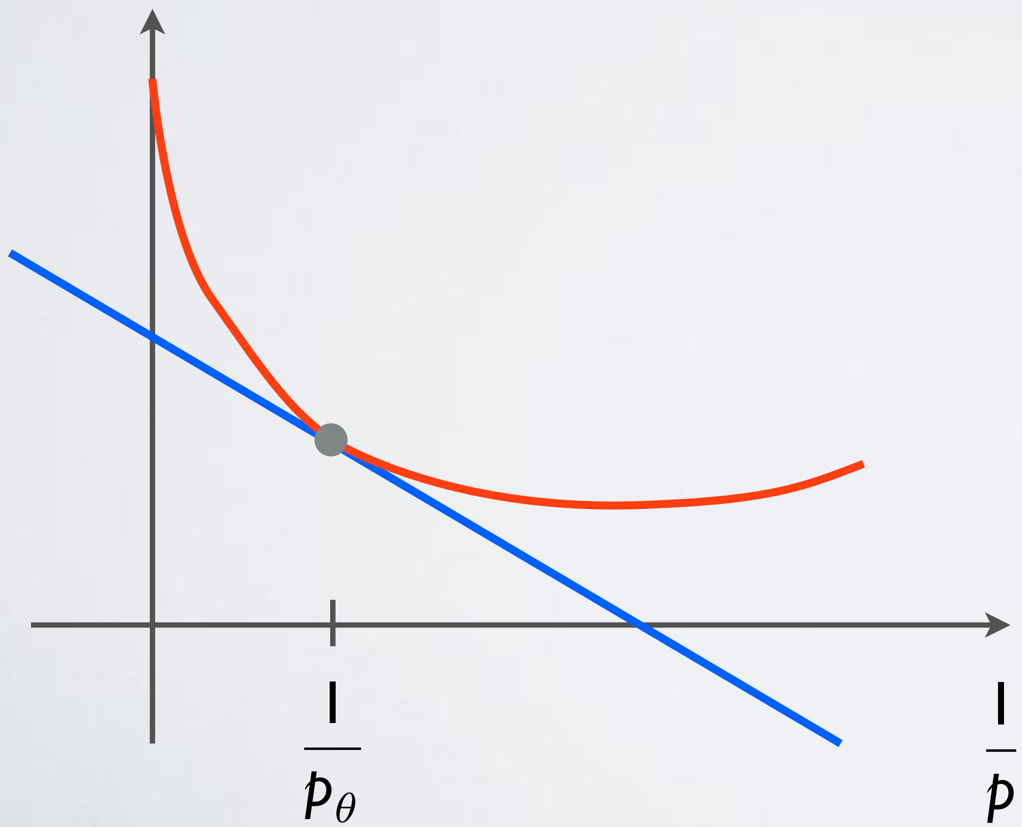
$$\|\phi_\lambda\|_{p_0} \leqslant M_0\, e^{c\,\mathrm{Re}\lambda} \qquad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{1-\theta} \cdot M_1^\theta$$

$$\Rightarrow$$

$$\|\phi_1\|_{p_1} \leqslant M_1 \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

# cf. Riesz-Thorin

| *duality* | ⇝ | *log-convexity* |
|---|---|---|
| change $p$ | | freeze $p$ |
| subharmonic | | harmonic |
| Hadamard | | Harnack |




$$\log \|\phi_\theta\|_p \geqslant A \cdot \frac{1}{p} + B$$

# Proof of the lemma

$$\log \|\phi_\lambda\|_p \geqslant A \cdot \frac{1}{p} + B(\lambda)$$

non-vanishing ⇝ harmonic

$$A \cdot \frac{1}{p_0} + B(\theta) \leqslant \theta \left( A \cdot \frac{1}{p_0} + B(1) \right)$$

Harnack

$$\log \|\phi_\theta\|_{p_\theta} = A \cdot \frac{1}{p_\theta} + B(\theta) = A \cdot \frac{1}{p_0} + B(\theta) + \theta \cdot A \left( \frac{1}{p_1} - \frac{1}{p_0} \right)$$

$$\leqslant \theta \left( A \cdot \frac{1}{p_1} + B(1) \right) \leqslant \theta \log \|\phi_1\|_{p_1} \leqslant 0 \qquad \square$$

Proof of main thm: $p \geq 2$

$f_{\bar z} = \mu f_z$ $\quad |\mu(z)| \leq \frac{1}{p-1} \chi_{\mathbb{D}}(z)$ $\quad f(z) = z + \mathcal{O}(1/z)$

want: $\frac{1}{\pi} \int_{\mathbb{D}} \left(1 - p \frac{|\mu(z)|}{1+|\mu(z)|}\right) |Df(z)|^p \, dm(z) \leq 1$

$p=2$ $\quad \frac{|Df(z)|^2}{K(z,f)} = J(z,f)$ null-Lagrangian ✓

$p=\infty$ $\quad K=1$ $\quad f(z)=z$ ✓

$\lambda \in U = \{\operatorname{Re} \lambda < 1/2\}$ $\quad f \leadsto f^\lambda$

$f^\lambda_{\bar z} = \mu_\lambda f^\lambda_z$

$f_{\frac{1}{p}} = f$

$f_0 = id$

$0 \quad \frac{1}{p}$

$U \quad 1/2$

$$\mu_\lambda(z) = \alpha_\lambda(z) \cdot \frac{\mu(z)}{|\mu(z)|} \qquad \text{"} \frac{0}{0} = 0 \text{"}$$

$$\frac{\alpha_\lambda(z)}{1+\alpha_\lambda(z)} = \lambda \cdot \underbrace{p\,\frac{|\mu(z)|}{1+|\mu(z)|}}_{\leq 1} \implies \|\mu_\lambda\|_\infty = \|\alpha_\lambda\|_\infty < 1$$

$$\mu_0 \equiv 0$$

$$\mu_{\frac{1}{p}} = \mu$$

$$\varphi_\lambda(z) = f_z^\lambda(z) + \frac{|\mu(z)|}{\mu(z)} f_{\bar z}^\lambda(z) \neq 0$$

$$\frac{J(z,f^\lambda)}{|\varphi_\lambda(z)|^2} = \frac{1-|\mu_\lambda(z)|^2}{|1+\alpha_\lambda(z)|^2} = 1 - 2\,\mathrm{Re}\,\frac{\alpha_\lambda}{1+\alpha_\lambda} \geq 1 - p\,\frac{|\mu|}{1+|\mu|} = w$$

- $\frac{1}{\pi}\int_{\mathbb{D}} |\varphi_\lambda|^2 w \leq \frac{1}{\pi}\int_{\mathbb{D}} J(z,f^\lambda) = \frac{|f^\lambda(\mathbb{D})|}{\pi} \leq 1 \qquad (p=2)$
- $\varphi_0 \equiv 1 \qquad (p=\infty)$

Interpolation lemma gives: $\frac{1}{\pi}\int_{\mathbb{D}} |\varphi_{\frac{1}{p}}|^p w \leq 1$

$$|\varphi_{\frac{1}{p}}| = |Df| \qquad w = 1 - p\,\frac{|\mu|}{1+|\mu|} \quad \text{tr}$$

# Corollaries

*sharp integrability estimates*

*LlogL* integrability: $f(z) \in z + C_0^\infty(\Omega)$, *homeomorphism*,

$$\int_\Omega \left(1 + \log|Df(z)|^2\right) J(z,f) \leqslant \int_\Omega |Df(z)|^2. \quad \textit{Proof:} \quad \left.\frac{dB_p}{dp}\right|_{p=2} \quad \square$$

Müller: *LlogL* integrability under J(z,f)≥0

*Exp* integrability: $|\mu(z)| \leqslant \chi_D(z), \quad z \in \mathbb{C}$

$$\frac{1}{\pi}\int_D \left(1 - |\mu|\right) e^{|\mu| + \operatorname{Re} S\mu} \leqslant 1. \quad \textit{Proof:} \quad \left.\frac{dB_p}{dp}\right|_{p=\infty} \quad \square$$

quasiconvexity:

$$\mathcal{H}(A) = \frac{1}{2}\frac{|A|^2}{\det A} + \log\left(\det A\right) - \log|A|, \qquad \det A > 0$$

# Mappings of integrable distortion

cf. Koskela-Onninen

**Corollary 4.3.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain, and suppose* $h \in \mathscr{W}^{1,1}_{loc}(\Omega)$ *is a homeomorphism* $h : \overline{\Omega} \overset{\text{onto}}{\to} \overline{\Omega}$ *such that* $h(z) = z$ *for* $z \in \partial\Omega$. *Assume* $h$ *satisfies the distortion inequality*

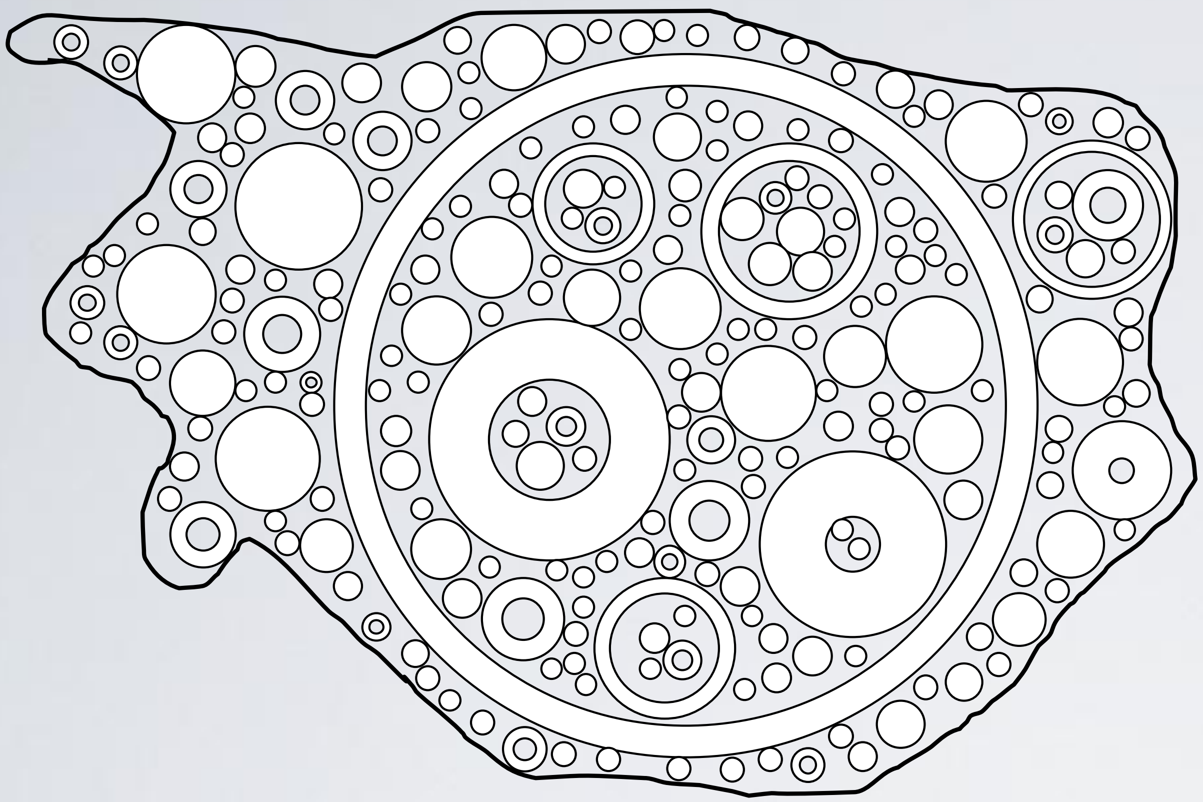
$$|Dh(z)|^2 \leqslant K(z)J(z,h), \qquad \text{a.e in } \Omega,$$

*where* $1 \leqslant K(z) < \infty$ *almost everywhere in* $\Omega$. *The smallest such function, denoted by* $K(z,h)$, *is assumed to be integrable. Then*

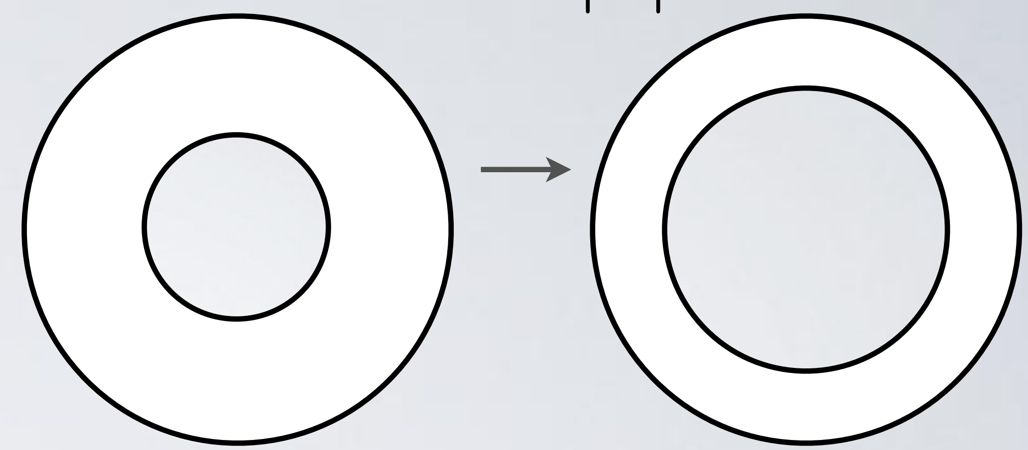
$$2\int_\Omega [\log |Dh(z)| - \log J(z,h)]\ dz \ \leqslant \ \int_\Omega [K(z,h) - J(z,h)]\ dz \tag{4.5}$$

*In particular,* $\log J(z,h)$ *is integrable. Again there is a wealth of functions, to be described in Section 5, satisfying* (4.5) *as an identity.*

# Many extremals



$$g(z) = \rho(|z|)\ \frac{z}{|z|}$$



*expanding*

$$\frac{\rho(t)}{t} \geqslant \dot{\rho}(t), \quad \rho(t) = o(t^{1-\frac{2}{p}})$$

$B_p$ linear on rank-one connections

Baernstein-Montgomery-Smith

$$\int_{B(0,R)} B_p(Dg) = \pi \int_0^R \left( \frac{[\,\rho(t)\,]^p}{t^{p-2}} \right)' \mathrm{d}t = \pi R^2$$

# Stretching vs Rotation

**harmonic dependece** "conjugate harmonic"

| stretching | rotation |
|---|---|
| **quasiconformal** | **bilipschitz** |
| Grötzsch problem | John's problem |
| Hölder exponent | rate of spiralling |
| $\log J(z,f) \in BMO$ | $\arg f_z \in BMO$ |
| higher integrability | exponential integrability |
| (linear) multifractal spectrum | |

# Complex exponents

Q: What complex exponents β can we take to make

$$f_z^{\beta} \in L^1_{loc} \quad ?$$


*foci* = "null-Lagrangians"

*eccentricity* = ellipticity coefficient = $k$

$$|\beta| + |\beta - 2| < \frac{2}{k}$$

controls

rotation & stretching

⇝ joint multifractal spectrum

# Outlook

- mappings of finite distortion (Eero)
- distortion of Hausdorff measures, removability (Ignacio)
- quasisymmetric maps, harmonic measure
- higher/even dimensions...