

6. soutěžní série – řešení

1. No. The value of both polynomials is a multiple of 3 at $x = 2$. The operations preserve multiples of 3, so the value of any polynomial obtained by the operations is a multiple of 3 at $x = 2$, but x isn't.

2. Denote a, b, c, d the entries of A and w_i, x_i, y_i, z_i the entries of B_i . Then the equality of determinants means $az_i + dw_i - cx_i - by_i = 0$. If there are four linearly independent solutions, then $(a, -b, -c, d)$ is orthogonal to $\mathbb{R}^4$, so it is the zero vector.

3. Denote X the set of points of discontinuity of the function f and assume for contradiction that X is uncountable. For a positive integer m denote

$$A_m = \{a \in X : f(x) > f(a) \text{ for all } x \text{ s.t. } 0 < \rho(x, a) < \frac{1}{m}\}$$

and

$$C_m = \{a \in X : f(x) < f(a) \text{ for all } x \text{ s.t. } 0 < \rho(x, a) < \frac{1}{m}\}.$$

Let $a \in X$. Obviously, if $\lim_{x \rightarrow a} f(x) > f(a)$, then $a \in A_m$ for some m and if $\lim_{x \rightarrow a} f(x) < f(a)$, then $a \in C_m$ for some m . So, these countably many sets cover the set X and therefore one of them is uncountable. Let us fix M s.t. A_M is uncountable (WLOG, C_M uncountable would be similar). Let Y be a countable dense subset of S . Then union of the balls $B(y, \frac{1}{2M})$ over $y \in Y$ is the whole of S . So, one of these balls (denote it B) contains at least two points from A_M . Let $a, b \in A_M \cap B$, $a \neq b$. Since $a \in A_M$ and $\rho(a, b) < \frac{1}{M}$ we have $f(b) > f(a)$. On the other hand, $b \in A_M$ and $\rho(b, a) < \frac{1}{M}$, therefore $f(a) > f(b)$. This is a contradiction.

4. He can. He sorts the numbers by their distance from $\frac{1}{2}$ and shows the second farthest. Denote it by x and WLOG $x < \frac{1}{2}$ (we ignore cases with zero probability). Lumír, even if he knows Karel's strategy, knows that the farthest number is either in the interval $(0, x)$, in which case the correct answer is "internal", or in the interval $(1 - x, 1)$, in which case the correct answer is "external". The two intervals have the same length, so his guess is fifty-fifty.

