

5. soutěžní série – řešení

1. For even n , the constant substrings will have an even length if and only if $a_{2k-1} = a_{2k}$ for each $k = 1, \dots, n/2$ (a_k denotes the result of the k -th toss). The probability is evidently $2^{-n/2}$. For odd n , it suffices to show that the number of possible tosses with strings of odd length is $Q(n) = 2F_n$. Certainly $Q(1) = 2$ and $Q(2) = 2$, and the recurrence $Q(n) = Q(n-1) + Q(n-2)$ holds (a string of length n has either the last two tosses the same, in which case we obtained it uniquely from a string of length $n-2$, or the last two tosses different, in which case we obtained it uniquely from a string of length $n-1$).

2. We easily guess that the zero polynomial $p(x) = 0$ satisfies the condition. For a non-zero polynomial, we must have polynomials of the same degree on both sides of the equation, from which we get the necessary condition $(\deg p)^2 = 2 + \deg p$, so we are looking for a quadratic polynomial. Since the equation $p(p(x)) = (x^2 + x + 1)p(x)$ holds for all real x , it also holds for complex numbers. If we substitute a (complex) root of p for x , then we get the condition $p(0) = 0$, so the polynomial we are looking for is of the form $ax^2 + bx$. After dividing by the polynomial $p(x)$, the given equation becomes $a(ax^2 + bx) + b = x^2 + x + 1 = a^2x^2 + abx + b$, which is satisfied if and only if $a = b = 1$. The polynomials $p(x) = 0$ and $p(x) = x^2 + x$ satisfy the condition.

3. (a) The sets $A = \{x \in \mathbb{Z}^n : \sum x_i \text{ is even}\}$ and $B = \{x \in \mathbb{Z}^n : \sum x_i \text{ is odd}\}$ evidently satisfy the condition — neither $D(A)$ nor $D(B)$ contains one.

(b) Let $A \cap B \neq \emptyset$, WLOG $0 \in A \cap B$ and $m = \sum x_i^2$, $\sqrt{m} \notin D(B)$. Then $x \in A$. Now for the sake of contradiction, let $\sqrt{n} \in D(\mathbb{Z}^n) \setminus D(A)$ and let y be such that $\sum y_i^2 = n$. Then $y \notin A$ and also $x + y \notin A$, but then $y, x + y \in B$, i.e., $\|(x + y) - y\| = \sqrt{m} \in D(B)$, a contradiction.

4. Let the limit exist and be equal to A . Without loss of generality $\lambda > \mu$; let $\alpha = \frac{\mu}{\lambda} < 1$. Then for $\varepsilon > 0$ there exists $\delta > 0$ such that for $|x| < \delta$ we have

$$A - \varepsilon < \frac{f(\lambda x) - f(\alpha \lambda x)}{x} < A + \varepsilon.$$

We can successively substitute $x\alpha^k$, $k = 1, 2, \dots, n$ and we obtain

$$\alpha^k(A - \varepsilon) < \frac{f(\alpha^k \lambda x) - f(\alpha^{k+1} \lambda x)}{x} < \alpha^k(A + \varepsilon),$$

summing these equations yields

$$\frac{1 - \alpha^{n+1}}{1 - \alpha}(A - \varepsilon) < \frac{f(\lambda x) - f(\alpha^{n+1} \lambda x)}{x} < \frac{1 - \alpha^{n+1}}{1 - \alpha}(A + \varepsilon),$$

for $n \rightarrow \infty$, due to continuity at zero, we have

$$\frac{A - \varepsilon}{1 - \alpha} \leq \frac{f(\lambda x) - f(0)}{x} \leq \frac{A + \varepsilon}{1 - \alpha}.$$

From here, it follows from the definition of the limit that $f'(0) = \frac{1}{\lambda} \frac{A}{1 - \alpha} = \frac{A}{\lambda - \mu}$.

Conversely, let f be differentiable at zero; then $\lim_{x \rightarrow 0} \frac{f(\lambda x) - f(0)}{\lambda x} = \lim_{x \rightarrow 0} \frac{f(\mu x) - f(0)}{\mu x} = f'(0)$, and from the theorem on the arithmetic of limits, it follows that $\lim_{x \rightarrow 0} \frac{f(\lambda x) - f(\mu x)}{x} = (\lambda - \mu)f'(0) = A$.

