

4. soutěžní série – řešení

1. Multiples of trace, that is $f(A) = k \cdot \text{tr } A = k \cdot \sum_{i=1}^n A_{i,i}$. Denote the basis matrices by $E_{i,j}$ (1 at i, j and 0 elsewhere). We know that f is defined by its values on the basis. If $i \neq j$, then $f(E_{i,j}) = f(E_{i,j}E_{j,j}) = f(E_{j,j}E_{i,j}) = f(0) = 0$. And $f(E_{i,i}) = f(E_{i,j}E_{j,i}) = f(E_{j,i}E_{i,j}) = f(E_{j,j})$. Multiples of trace are thus the only possibility and it is easy to prove by definitions that $\text{tr } AB = \text{tr } BA$.

2. Marked points divide the circle to 100 arcs. Pokus finds the longest arc A and erases its clockwise endpoint. The arc A and its clockwise neighbour B are merged into a new single longest arc AB . Then comes Hokus, locates the longest arc and marks the semicircle whose clockwise endpoint coincides with the clockwise endpoint of the largest arc (which is AB). Since B wasn't originally the single longest arc, it was not longer than a semicircle, so its counterclockwise endpoint, which was erased, indeed lied in the semicircle.

3.

We show $k = 4m^2$ works. Assume for contradiction that $x^2 + y^2 + m = c \cdot 4m^2$. Then $x^2 + y^2 = m(4cm - 1)$. Since $4cm - 1 \equiv -1 \pmod{4}$, there exists a prime $p \equiv -1 \pmod{4}$ s.t. p -valuation $v_p(4cm - 1)$ is odd. In particular $p \mid 4cm - 1$, so p does not divide m . Therefore, $v_p(m(4cm - 1)) = v_p(x^2 + y^2)$ is odd. This is however contradiction with "Sum of two squares" theorem.

4. Define $b_n = \frac{1}{n} \sum_{i=1}^n a_i$ and $s_n = \sum_{i=1}^n \frac{a_i}{i}$ for $n \geq 0$ and let $s := \lim_{n \rightarrow \infty} s_n$. We have

$$b_n = \frac{1}{n} \sum_{i=1}^n i(s_i - s_{i-1}) = s_n - \frac{1}{n} \sum_{i=0}^{n-1} s_i$$

But $s_n \rightarrow s$ and also $\frac{1}{n} \sum_{i=0}^{n-1} s_i$ converge to s (Cesaro sum) for $n \rightarrow \infty$. So b_n converges to 0 for $n \rightarrow \infty$.

