3. soutěžní série – řešení

1. Suppose that the minimal length from city A is m_A. Any city B then lies on a route of length m_A. To visit all cities starting in B, one can first go to the closer end of the route from A, which is in a distance at most $m_A/2$, and the go along the route. One can thus visit all cities starting from B in length $\leq 3/2 \cdot m_A$.

2. Multiply by -1 all the rows with -1 in the first column. Then subtract the first row from all the other rows. After this, there is only one non-zero entry in first column, it is 1 in the first row. Moreover, all entries of the sub-matrix consisting of rows 2, …, n and columns 2, …, n are 0 or ± 2. We are done.

3. We first prove that even inner segments are divided to equal parts. Denote the vertices of the grid $P_{i,j}$ for $i, j \in \{0, \dots, N\}$ and areas of the tiles $A_{i,j}$ for $i, j \in \{1, \dots, N\}$. We have

$$P_{0,j} = \frac{(N-j)P_{0,0} + jP_{0,N}}{N}$$

and

$$P_{N,j} = \frac{(N-j)P_{N,0} + jP_{N,N}}{N}.$$

Indeed, we can check that the point $\frac{(N-j)P_{i,0}+jP_{i,N}}{N}$, which lies on the segment $P_{i,0}P_{i,N}$ in the correct ratio, lies on $P_{0,j}P_{N,j}$, so it is in fact $P_{i,j}$. Similarly along the other axis.

Then areas of triangles $P_{i,j}P_{i,j+1}P_{i-1,j}$ and $P_{i,j}P_{i,j+1}P_{i+1,j}$ are equal – they have equal side length $P_{i,j}P_{i-1,j} = P_{i,j}P_{i+1,j}$ and equal height. Take the four tiles that share a vertex $P_{i,j}$ and divide each into two triangles by the diagonal from $P_{i,j}$. Then we can use the previous observation four times and we get that $A_{i,j} + A_{i+1,j+1} = A_{i,j+1} + A_{i+1,j}$. By using this equality repeatedly we get $A_{i,j} + A_{i+k,j+l} = A_{i,j+l} + A_{i+k,j}$. This allows us to transform any choice (permutation) of tiles to any other, and since N permutations can cover the quadrilateral, the proof is complete.

4. The unknown function has the form

$$f(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}, \qquad a_n, b_m \neq 0.$$

There exists a sequence $\{x_j\}_{j=1}^{\infty}$ of points for which $f(x_j) \in \mathbb{Z}$. In particular, $q(x_j) \neq 0$, so there is no issue with division by zero at the points x_j. Without loss of generality, we may assume that $\lim_{j\to\infty} |x_j| = \infty$.

For now, assume that all coefficients of $p(x)$ and $q(x)$ are rational. By polynomial division, we obtain $f(x) = s(x) + \frac{r(x)}{q(x)}$, where $\deg r < \deg q$. Multiply the equation by a number $N \in \mathbb{N}$ such that the polynomial $Ns(x)$ has integer coefficients. Then the left-hand side $Nf(x_j) - Ns(x_j) = \frac{Nr(x_j)}{q(x_j)}$ is an integer, while the right-hand side tends to zero due to the higher degree of the denominator. Therefore, $r(x_j) = 0$ for all sufficiently large j. Hence, r is identically zero and $f = s$ is a polynomial.

It remains to justify that the coefficients $a_0, \dots, a_n, b_0, \dots, b_m$ are rational. For a fixed x_j, from the identity $\sum_{i=0}^{n} a_i x_j^i = f(x_j) \sum_{i=0}^{m} b_i x_j^i$ we obtain a linear equation in $n+m+2$ unknowns. There are infinitely many such equations, and we already know that a nontrivial solution exists. From this, it follows that there exist polynomials p_1, q_1 with rational coefficients such that $\frac{p_1(x_j)}{q_1(x_j)} = f(x_j) = \frac{p(x_j)}{q(x_j)}$ for all $j \in \mathbb{N}$. The polynomial $p_1(x)q(x) - p(x)q_1(x)$ has infinitely many roots, hence it is identically zero. Therefore, $f(x) = \frac{p_1(x)}{q_1(x)}$ for all $x \in \mathbb{R}$, except possibly for finitely many zeros of the polynomials q and q_1.

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