2. soutěžní série – řešení

1. Induction: there's an empty row, we move it to the top and then we move a column containing at least one marked field to the left. Now $n-2$ marked fields are left in the $(n-1)\times(n-1)$ bottom right subtable. Hence, we proceed by induction hypothesis, noting that we can't get any marked field to the first row by subsequent operations.

2. First note that f is injective: if $f(a)=f(b)$, then

$$a+2026=f(2f(a))=f(2f(b))=b+2026,$$

hence $a=b$. Now consider $f(1), f(3), f(5), \dots, f(4053)$. These must be distinct, so we have 2027 distinct numbers. Therefore from $1, 3, \dots, 4053$ we can choose k such that $f(k)\geq 2027$. But then

$$f(k)=(f(k)-2026)+2026=f(2f(f(k)-2026)),$$

so by the injectivity of f we obtain $k=2f(f(k)-2026)$, which is a contradiction because k is odd. Hence no function f satisfying the conditions exists.

3. Let $a_n=\cos(n\pi x)$ and $b_n=\cos(n\pi y)$. Then

$$(a_n+b_n)^2+(a_n-b_n)^2=2a_n^2+2b_n^2=a_{2n}+1+b_{2n}+1=2+(a_{2n}+b_{2n}),$$

since $\cos 2z=2\cos^2 z-1$. Thus, if $\{a_n+b_n:\ n\in\mathbb{N}\}$ is finite, then $\{a_n-b_n:\ n\in\mathbb{N}\}$ is also finite. But then the sets $\{a_n:\ n\in\mathbb{N}\}$ and $\{b_n:\ n\in\mathbb{N}\}$ are finite as well, because

$$a_n=\frac{1}{2}((a_n+b_n)+(a_n-b_n)), \qquad b_n=\frac{1}{2}((a_n+b_n)-(a_n-b_n)).$$

From this it follows that $x, y\in\mathbb{Q}$.

4. By coloring the points with coordinates satisfying $x\equiv 2y \pmod 7$, $x\equiv 2y+3 \pmod 7$, we color at most $14\left(\frac{n+6}{7}\right)^2$ points, and the only uncolored segments that remain are horizontal segments of lengths 2 and 3, which cannot be completed into uncolored squares. On the other hand, every 1×1 square contains at least one colored point.

We say that if a square contains k colored points, each point contributes $\frac{1}{k}$. Summing the contributions over the squares clearly gives n^2. For every point inside the grid there is some colored point in the surrounding 2×2 square, and therefore it can contribute at most $1+1+1+\frac{1}{2}$. A point on the boundary of the grid can contribute at most 2. Summing the contributions over the colored points we obtain $n^2\leq\frac{7}{2}\,l(n)$. Altogether $\frac{2}{7}\leq\frac{l(n)}{n^2}\leq\frac{2}{7}\frac{(n+6)^2}{n^2}$, and the limit follows by the squeeze theorem.