1. contest series – solution

1. Let us denote $p(2025)$ by a and observe that $2025 = 81 \cdot 25$. Obviously any number ending with 25 is divisible by 25, so divisibility by 81 is necessary and sufficient condition. So, a has to be a multiple of 9. Therefore, if a had just 1 digit, we would have $a = 9$ but 20925 is not divisible by 81. We show there is a unique two-digit number a which satisfies the condition.

Let $M = 20 \cdot 10^4 + 100a + 25$ and $a = 9b$. We have $81 \mid M = 200025 + 900b$, which is equivalent to $81 \mid 36 + 9b$ (compute modulo 81). Since $a = 9b$ is a two-digit number, we have $46 \leq 36 + 9b \leq 135$. Due to these bounds, $81 \mid 36 + 9b$ is equivalent to $36 + 9b = 81$, i.e. $9b = 45 = a$ is the only 2-digit solution.

2. Assume that the circle in the problem is the unit circle in the complex plane with center at the origin. Let ω denote $e^{\frac{i\pi}{3}}$. Since triangles ABO, CDO, and EFO are equilateral, we have $B = \omega A$, $D = \omega C$, and $F = \omega E$. Then $K = \frac{B+C}{2} = \frac{\omega A+C}{2}$, and similarly $L = \frac{\omega C+E}{2}$ and $M = \frac{\omega E+A}{2}$. Then

$$K - L = \frac{\omega A + (1-\omega)C - E}{2} \quad \text{and} \quad M - L = \frac{A - \omega C + (\omega - 1)E}{2}.$$

Since $1 - \omega = -\omega^2$ and $\omega(\omega - 1) = \omega^3 = -1$ we have $K - L = \omega(M - L)$, which exactly means that triangle KLM is equilateral.

3. We deduce the result from the inequalities

$$\begin{aligned} r(XY) &\leq \min\{r(X), r(Y)\}, \\ r(XY) &\geq r(X) + r(Y) - n. \end{aligned}$$

To prove them, we use that $r(A) = \dim(\operatorname{im} A)$. The first inequality is well known: it's because $\operatorname{im} XY \subseteq \operatorname{im} X$, so we have $\dim(\operatorname{im} XY) \leq \dim(\operatorname{im} X)$ and in a transposed way for Y. The second one is due to the fact that $\dim(\ker XY) \leq \dim(\ker X) + \dim(\ker Y)$ (take a basis of $\ker Y$, expand it to a basis of $\ker XY$, map the new vectors by Y – they must fall into $\ker X$ and stay independent). Then we use $\dim(\operatorname{im} A) + \dim(\ker A) = n$ to get the inequality.

Now suppose WLOG that $\min\{r(X), r(Y)\} = r(X)$. If $r(X) \leq n/2$, then we get $0 \leq r(XY), r(YX) \leq n/2$ by the first inequality and we are done. If $r(X) \geq n/2$, then $r(XY) \geq r(X) + r(Y) - n \geq r(Y) - n/2 \geq r(YX) - n/2$ and the other way around.

4. If $\liminf na_n = 0$, we construct a sequence b_n as follows. Choose n_1 such that $n_1 a_{n_1} < \frac{1}{2}$ and for $k > 1$ choose $n_k > n_{k-1}$ such that $n_k a_{n_k} < \min\{n_{k-1} a_{n_{k-1}}, 2^{-k}\}$. Define $b_n = a_{n_k}$ for n satisfying $n_{k-1} < n \leq n_k$. Then (b_n) is nonincreasing and

$$\sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} a_{n_k}(n_k - n_{k-1}) \leq \sum_{k=1}^{\infty} n_k a_{n_k} \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Conversely, if $\liminf na_n = c > 0$, there exists N such that for all $n \geq N$ we have $na_n > \frac{c}{2}$. Let $b_k \geq a_k$ for infinitely many k. From these indices choose a subsequence $(n_k)_{k=1}^{\infty}$ satisfying $n_1 > N$ and $n_k > 2n_{k-1}$ for $k > 1$. Then

$$\sum_{n=1}^{\infty} b_n \geq \sum_{n=n_1}^{\infty} b_n \geq \sum_{k=2}^{\infty} b_{n_k}(n_k - n_{k-1}) > \sum_{k=2}^{\infty} \frac{1}{2} b_{n_k} n_k \geq \sum_{k=2}^{\infty} \frac{1}{2} a_{n_k} n_k \geq \sum_{k=2}^{\infty} \frac{c}{4} = \infty.$$