Mathematics Behind Recommender Systems

Academia meets industry

Pavel Kordík

Recombee & FIT CTU Prague

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Introduction

Recommendations are everywhere

Every time you see:

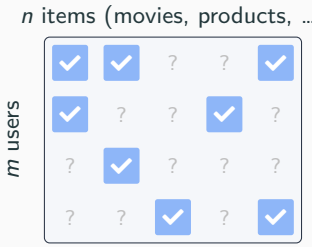
- *"You might also like..."* on Netflix
- *"Customers also bought..."* on Amazon
- *"Discover Weekly"* on Spotify
- *"For You"* in a news feed

... there is a recommender system behind it.

The core challenge:

Millions of items, millions of users.

How to find the *right* item for *each* user?



Interaction matrix: most entries are **unknown**

Recommendation and search are converging

Both solve the **same core problem**: produce a ranked list of items.

Search	Recommendation
User provides a **query**	System infers intent from **user profile**
$\rightarrow$ ranked relevant documents	$\rightarrow$ ranked personalized items
Google, Elasticsearch, e-shop search	Netflix, Spotify, news feeds

The convergence:

- Both optimize **ranking metrics** (not classification)
- Modern search is personalized; modern RecSys supports queries
- Shared infrastructure: embeddings, ANN indices, learning-to-rank

Recombee

Recommendation & Search as a Service

- Founded in 2015 with my PhD students
- Bootstrapped to $6M ARR
- 500+ customers in 40+ countries
- Billions of interactions processed daily
- Personalized recommendation, semantic search, real-time analytics

Customers: The Telegraph, DAZN, Pepper, Prima, Český rozhlas, Mediavine, Neura, 9GAG, and many more across media, e-commerce, video, and marketplaces.

prg.ai Minor students at Recombee HQ, Václavské náměstí 1, Prague

Recombee Lab @ FIT CTU

Joint university-industry research lab at the Dept. of Applied Mathematics, FIT CTU in Prague.

Focus:

- Linear autoencoders & matrix factorization
- Bandit algorithms & exploration
- Evaluation methodology
- NLP for recommendations

Collaborations: Ladislav Peška group at MFF UK, TU Kaiserslautern, SMU Singapore

🔗 `kam.fit.cvut.cz/recombee-lab`

Today's program

1. Linear autoencoders

From EASE to ELSA, VASP, and Sparse ELSA

2. Matrix factorization

SVD, ALS, and describing taste with numbers

3. Bandit algorithms

Exploration vs. exploitation

4. Sequential & language models

ReALM and beeFormer

5. Evaluation & production

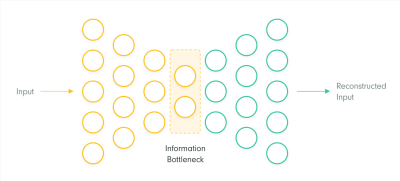
Ranking metrics, offline bias, deployment

6. Research overview & RecSys 2025

Linear Autoencoders

What is an autoencoder?

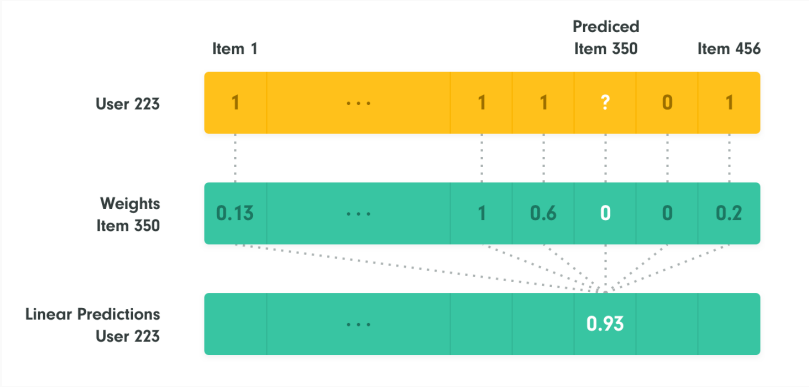
An **autoencoder** learns to compress and reconstruct data through an information bottleneck:



- Input = what a user has consumed (n-dimensional binary vector)
- Bottleneck = compressed "taste profile"
- Output = predicted scores for *all* items
- A linear autoencoder uses no activation functions – just matrix multiplication

recombee.com/blog – Linear Methods and Autoencoders

Linear prediction: How it works



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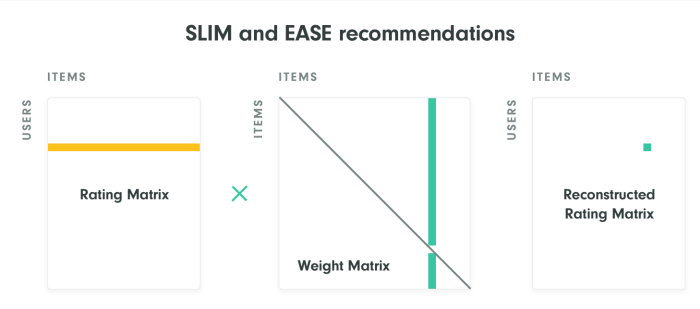
SLIM and EASE: The shallow approach

No hidden layers – just an $n \times n$ weight matrix $\mathbf{W}$:



Rating Matrix $\times$ Weight Matrix $=$ reconstructed matrix. The diagonal is zeroed out, so an item cannot predict itself.

recombee.com/blog – Linear Methods and Autoencoders

EASE: Embarrassingly Shallow Autoencoders

Optimization: Find $\mathbf{W}$ that best reconstructs the data:

$$\mathbf{W}^* = \arg\min_{\mathbf{W}} \|\mathbf{X} - \mathbf{X}\mathbf{W}\|_F^2 + \lambda\|\mathbf{W}\|_F^2, \quad \text{diag}(\mathbf{W}) = \mathbf{0}$$

Closed-form solution (one step!)

$$\mathbf{W}^* = \mathbf{I} - \hat{\mathbf{P}} \cdot \text{diag}(\hat{\mathbf{P}})^{-1}, \qquad \hat{\mathbf{P}} = (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}$$

- Closed-form solution from one matrix inverse
- Negative weights capture dissimilarity
- **Problem:** Inverting an $n \times n$ matrix costs $O(n^3)$

Steck, EASE, WWW 2019

The scalability wall

How bad is $O(n^3)$ in practice?

Some Recombee clients have **hundreds of millions of items** in their catalogs.

Catalog size n	Matrix W	Inverse time (approx.)
10,000	$10^4 \times 10^4$	seconds
100,000	$10^5 \times 10^5$	minutes
1,000,000	$10^6 \times 10^6$	days
10,000,000	$10^7 \times 10^7$	impossible (4 PB RAM)
100,000,000	$10^8 \times 10^8$	utterly impossible

Two approaches to break the wall:

1. **ELSA** (Recombee / FIT CTU): Low-rank factorization $\mathbf{W} = \mathbf{A}\mathbf{A}^\top - \mathbf{I} \to O(nd^2)$
2. **SANSA** (MFF UK + GLAMI): Sparse approximate inverse of the Gram matrix $\to$ $O(n \cdot \text{nnz})$

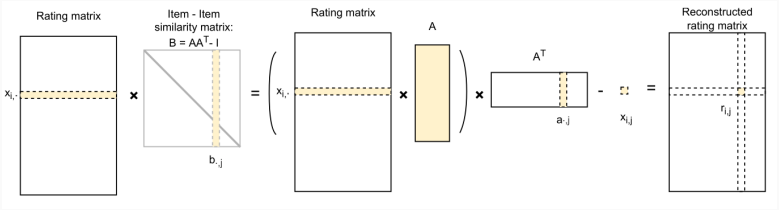
ELSA: Making it scalable

ELSA was developed jointly at **FIT CTU + Recombee**. **Key idea:** Instead of storing the full $n \times n$ matrix $\mathbf{W}$, represent each item as a d-dimensional embedding vector ($d \ll n$):

Each item i gets an embedding $\mathbf{a}_i \in \mathbb{R}^d$. Stacking all items: $\mathbf{A} \in \mathbb{R}^{n \times d}$.

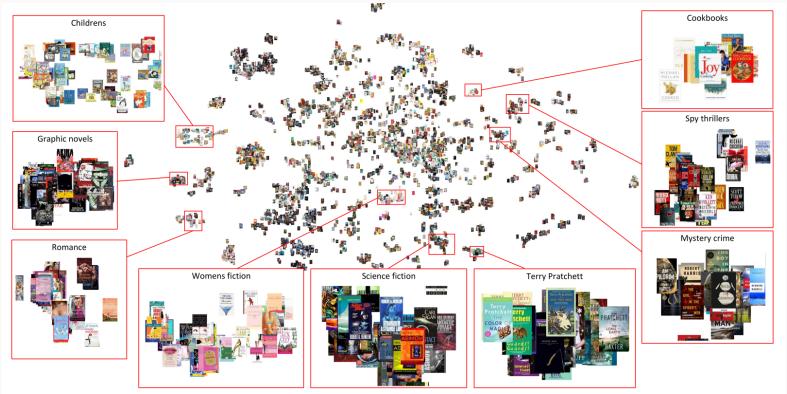
After row normalization, item similarity becomes cosine similarity:

$$W_{ij} = \frac{\mathbf{a}_i^\top \mathbf{a}_j}{\|\mathbf{a}_i\| \, \|\mathbf{a}_j\|} - \delta_{ij}$$



Vančura et al., ELSA, RecSys 2022

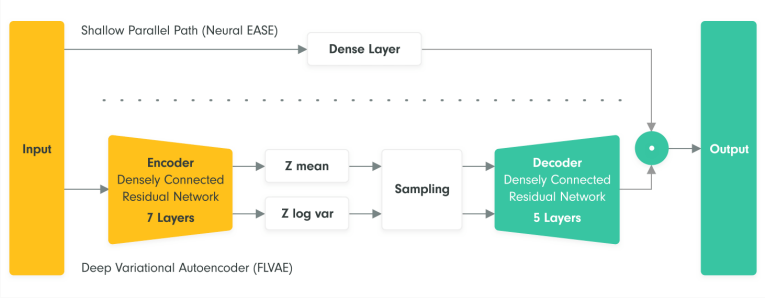
ELSA: What the embeddings look like



Vančura et al., ELSA, RecSys 2022 – Goodreads embeddings

VASP: Combining linear and deep paths

VASP combines the robustness of a linear model with the expressiveness of a deep generative model.

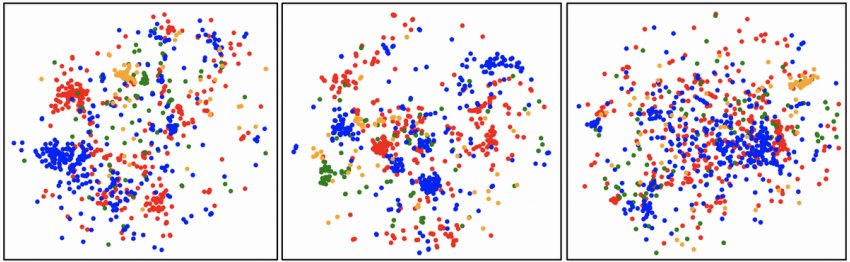


Linear and deep paths are combined via Hadamard $\odot$, so both must agree.

Vančura, Kordík, VASP, RecSys 2021

VASP: What each path captures

t-SNE of MovieLens embeddings, colored by genre:



Left: VASP output. **Center:** EASE (linear) – smooth patterns. **Right:** FLVAE (deep) – complex clusters. Colors: Horror, Children's, Westerns, Noir.

Vančura, Kordík, VASP, RecSys 2021

Sparse ELSA: Interpretable embeddings from interactions

Idea: Gradually prune ELSA embeddings during training:

$$\mathbf{A}_s^{(t)} = \mathcal{S}_{k_t}(\mathbf{A}^{(t)}), \qquad k_0 = d \xrightarrow{\text{exp. decay}} k_T = k$$

- $\mathcal{S}_{k_t}$ keeps only the k_t strongest coordinates and zeros the rest
- Training starts dense ($k_0 = d$) and gradually ends with only k active dimensions

Results (nDCG@100):

	Dense	**10× compr.**
Goodbooks	0.489	**0.491**
Netflix	0.395	0.393

At 10× compression, the loss is very small.

Sparse factors naturally correspond to semantic segments – without any metadata:

- Children's classics
- Detective fiction
- Sci-fi romance

Pure interaction data → interpretable structure.

Matrix Factorization

Intuition: The latent space of taste

Imagine: Each movie can be described by a few numbers – how much action, romance, humor...

And each viewer by the *same* numbers – how much they enjoy each.

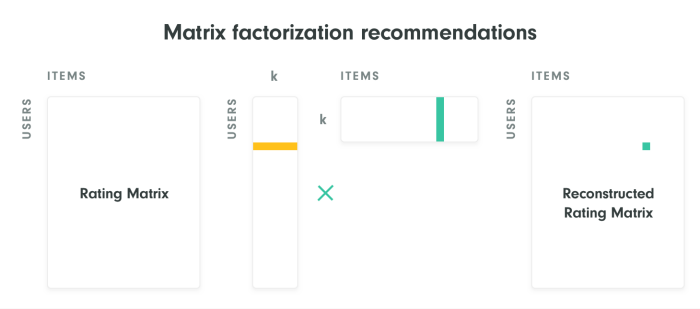
Matrix Factorization

$$\underbrace{\mathbf{X}}_{m \times n} \approx \underbrace{\mathbf{U}}_{m \times r} \cdot \underbrace{\mathbf{V}^\top}_{r \times n}$$

- $\mathbf{U}$ – user "profiles" in r-dimensional latent space
- $\mathbf{V}$ – item "profiles" in the same space
- $r \ll \min(m, n)$ – typically 50–200 dimensions

Prediction: $\hat{X}_{ij} = \mathbf{u}_i^\top \mathbf{v}_j$ (dot product = "compatibility")

Matrix factorization: Visual



Rating Matrix $\approx$ Users $(m \times k)$ $\times$ Items $(k \times n)$. Low-rank factors capture latent preferences and item characteristics.

recombee.com/blog – Linear Methods and Autoencoders

Finding the factorization: ALS

Alternating Least Squares – an elegant iterative approach:

1. **Fix items $\mathbf{V}$**, optimize users $\mathbf{U}$:
   $\mathbf{U}^* = (\mathbf{V}^\top \mathbf{V} + \lambda \mathbf{I})^{-1} \mathbf{V}^\top \mathbf{X}^\top$ (just linear regression!)
2. **Fix users $\mathbf{U}$**, optimize items $\mathbf{V}$:
   $\mathbf{V}^* = (\mathbf{U}^\top \mathbf{U} + \lambda \mathbf{I})^{-1} \mathbf{U}^\top \mathbf{X}$ (same, roles swapped)
3. Repeat until convergence.

Why ALS?

- Each step has a closed-form solution (no gradient descent)
- Parallelizable across rows / columns
- Works for implicit feedback (clicks instead of ratings)

Bandit Algorithms

The exploration–exploitation dilemma

Imagine a restaurant:

- **Exploitation:** Order your favorite steak (certainty)
- **Exploration:** Try a new Thai dish (might be great, might not)

Exploitation

Recommend what the model knows works: safe, but no new information.

Exploration

Try new / uncertain items: risky, but we learn.

Multi-armed bandit: A mathematical framework for decision-making under uncertainty. Named after a gambler facing a row of slot machines.

Thompson Sampling

Idea: For each item, maintain a probability distribution over its quality.

Algorithm

1. For each item i: sample $\theta_i \sim \text{Beta}(\alpha_i, \beta_i)$
   where $\alpha_i = \text{successes} + 1, \quad \beta_i = \text{failures} + 1$
2. Recommend the item with the highest sample: $i^* = \arg\max_i \theta_i$
3. Observe outcome, update α_{i^*} or β_{i^*}

Why it works:

- Uncertain items $\rightarrow$ wide distribution $\rightarrow$ sometimes high sample $\rightarrow$ exploration!
- Reliable items $\rightarrow$ narrow distribution around high value $\rightarrow$ exploitation!
- Natural balance without parameter tuning

BMAB: What if popularity changes over time?

Problem: Standard bandits assume quality is *constant*. In practice: viral videos, breaking news, seasonal trends...

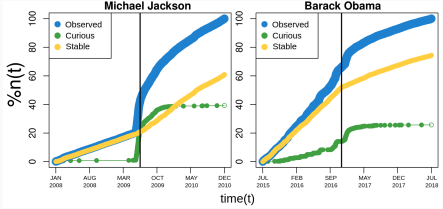
Two types of audience

1. **Loyal:** Stable interest (fans)
2. **Curious:** Burst activity (viral wave)

Model: Mixture of two Poisson processes

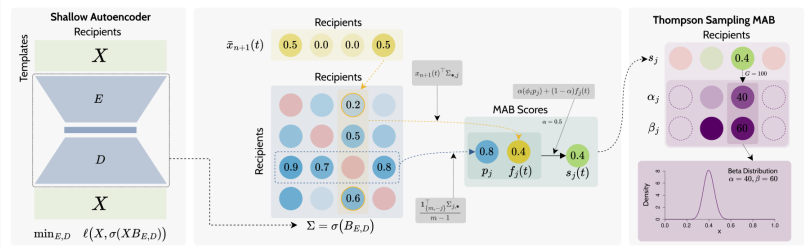
$$N(t) \sim \text{Poi}(\lambda_{\text{loyal}}) + \text{Poi}(\lambda_{\text{curious}}(t))$$

Burst detected $\rightarrow$ reset priors $\rightarrow$ rapid adaptation.



Stable + bursty audiences in real traffic.

Active Recommendation for Email Outreach



We do this to reduce spam: send fewer emails, keep open rate high, and still learn which recipients respond to new templates. Shallow autoencoders score cold-start templates and Thompson Sampling adaptively chooses recipient batches in a 15M-interaction outreach platform.

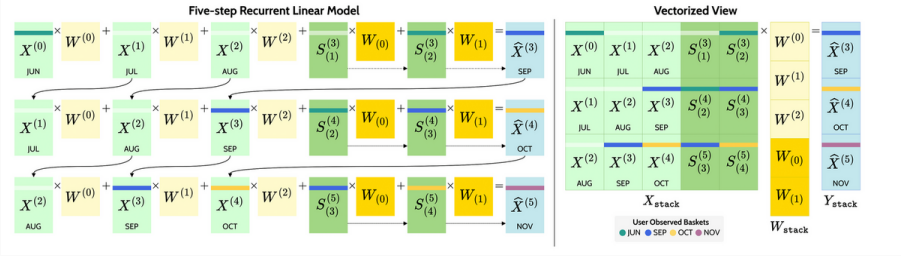
Zid, Alves, Kordík — CIKM 2025

Sequential & Language Models

ReALM: Predicting the next shopping basket

Rows = users, columns = visits. User 1 always buys fruit; diapers grow S→M→L.

ReALM: The model



- Autoregressive linear model
- Captures temporal dependencies
- Closed-form analytical solution – no SGD

Where LSTM/Transformers train for hours, ReALM converges in **seconds**.

Zmeškalová et al., ReALM, RecSys 2025

The cold-start problem

What if we know nothing about an item?

A new movie in the catalog – nobody has seen it. Collaborative filtering fails.

Solution: We have *text descriptions*! Title, genre, synopsis...

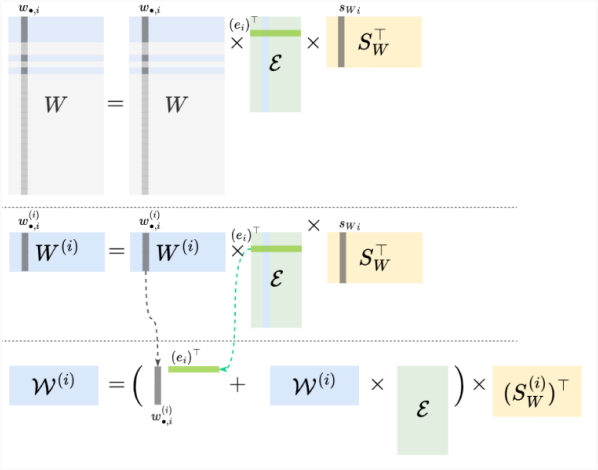
Sentence Transformers

Models that convert text into numerical vectors (embeddings).
Similar texts $\rightarrow$ similar vectors.

But: Semantic similarity $\neq$ recommendation similarity.
"Star Wars" and "Documentary about George Lucas" are semantically close, but fans of one may not want the other.

IndAE: Autoencoders for cold-start items

Problem: EASE/ELSA cannot recommend new items with zero interactions. We still want to score a brand-new item from its metadata before anyone clicks on it.



beeFormer: Architecture

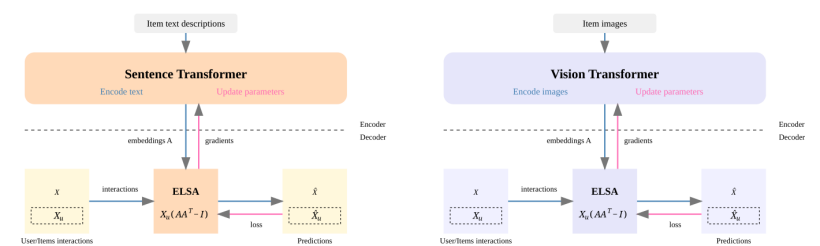


Fig. 1. BeeFormer is a Transformer-based Architecture for Vision-Language Tasks. It can directly use raw text, process it with a Vision Encoder, and handle multiple modal and textual inputs through either a cross-modal or text sequence processor.

Transformer encodes item text/images into embeddings $\mathbf{A}$. ELSA loss $\mathcal{L} = \|\mathbf{X} - \mathbf{X}(\mathbf{A}\mathbf{A}^\top - \mathbf{I})\|_F^2$ measures reconstruction quality. Gradients flow back into the Transformer – it learns to predict *user behavior*, not just semantic similarity.

beeFormer: Results

Zero-shot transfer

Trained on MovieLens + Amazon, tested on Goodbooks-10K:

Model	Recall@20
Semantic ST	0.1146
beeFormer	**0.2649**

+131% improvement

Cold-start (ML-20M)

New items with no interactions:

Model	Recall@20
Heater	0.3114
beeFormer	**0.4630**

+49% improvement

Multi-domain training yields +8% – knowledge accumulates across domains.

SHIELD: Making semantic search safe

Problem: Semantic search understands meaning, so harmful intent is harder to catch than with keywords alone.

SHIELD: 3-stage framework

1. **Generate** realistic harmful/sensitive queries with LLMs
2. **Filter** high-quality examples with a reward model
3. **Classify** queries in the moderation pipeline

Results:

Method	Accuracy
BM25 keyword	93.2%
Semantic + FAISS	96.5%
MoralBERT	**98.4%**

Open source:

`flpspacek/SHIELD`

`spacefi1/moralBERT`

Applicable beyond marketplaces too.

Spaček, Vančura, Kordík — UMAP 2025

Evaluation & Production

How do we evaluate a ranked list?

Unlike classification (accuracy) or regression (RMSE), ranking cares about order and position.

Key metrics

- **Precision@K:** Fraction of top-K items that are relevant
- **Recall@K:** Fraction of all relevant items found in top-K
- **NDCG@K** (Normalized Discounted Cumulative Gain):
  Rewards relevant items appearing *higher* in the list

NDCG:

$$\text{DCG@K} = \sum_{i=1}^{K} \frac{\text{rel}_i}{\log_2(i+1)}, \qquad \text{NDCG@K} = \frac{\text{DCG@K}}{\text{IDCG@K}} \in [0,1]$$

Position 1 contributes $\frac{1}{\log_2 2} = 1.0$; position 10 only $\frac{1}{\log_2 11} = 0.29$.

The problem with offline evaluation

In theory: Compute NDCG on held-out data, pick the best model, deploy.

In practice: This often fails. Why?

The core bias: offline data reflects the *old* system

We only observe interactions with items the **previous recommender** chose to show. A new model that recommends *different* items looks bad offline – those items were never shown, so there is no positive signal for them.

This creates a systematic bias toward exploitative algorithms:

- Models that mimic the old system score well offline
- Models that **explore** new items score poorly – even if users would love them
- Popular items dominate the test set $\rightarrow$ popularity-chasing wins offline

Our contribution: Better offline–online correlation

Solution: Leave-*Last*-One-Out + popularity penalization

$$\text{recall}^{\beta}_{\text{LLOO}} \quad \text{with weights } p(i)^{-\beta}, \quad \beta \approx 0.30$$

Method	MSR
Standard LOO-CV	12.9%
LLOO + β	**34.3%**

Leaving out the *last* interaction respects chronology, so evaluation reflects what the user did next rather than a random held-out event. Popularity penalization reduces the bias toward models that only repeat what the old system already showed.

Kasalický, Alves, Kordík — evalRS @ KDD 2023

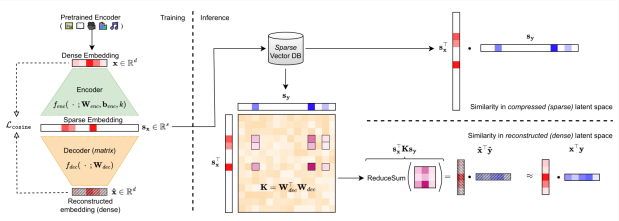
RepSys: Interactive evaluation framework

Open-source app for interactive comparison of recommendation models.

`cowjen01/repsys`

CompresSAE: How it works

We use CompresSAE because modern embedding models are accurate but too memory-hungry at production scale.

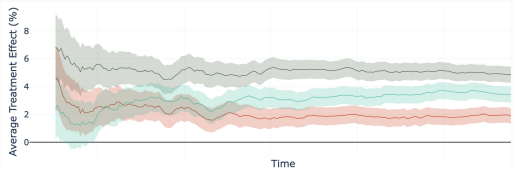


- Training learns a sparse code that reconstructs the dense embedding
- Inference compares sparse codes directly or via a kernel

Kasalický et al., CompresSAE, RecSys 2025

CompresSAE: Online A/B test results

Problem: 100M items $\times$ 768-dim = 307 GB memory for embeddings.



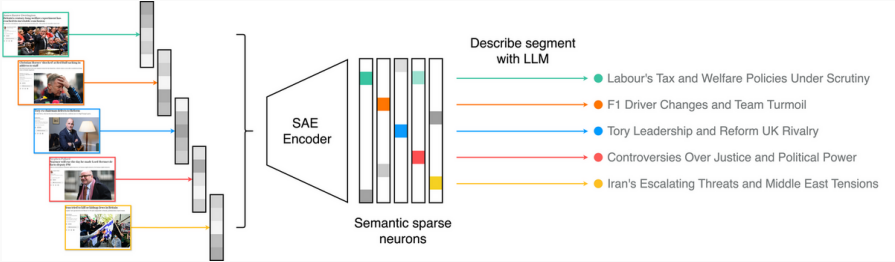
	CTR lift	Size
Dense 768	+4.86%	307 GB
CompresSAE	**+3.44%**	**26 GB**

12$\times$ compression, 1.35% CTR loss.

15 seconds on H100.

Kasalický et al., CompresSAE, RecSys 2025

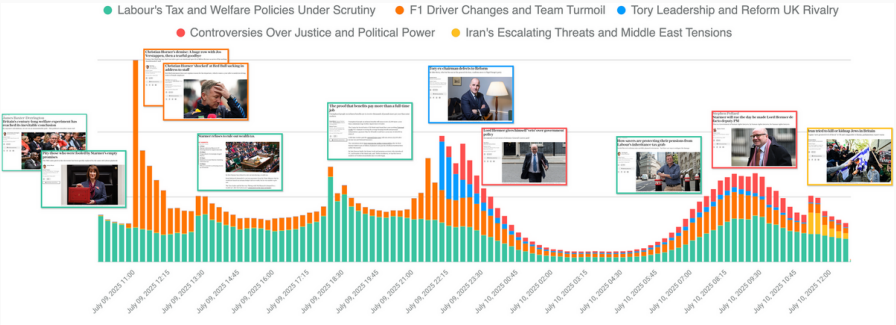
The Telegraph: Segment-aware editorial analytics



SAE encoder maps articles to **semantic sparse neurons**. An LLM describes each segment. Editors see which reader segments engage with which topics.

The Telegraph case study, INRA @ RecSys 2025

The Telegraph: Segment dynamics in real time



Content consumption by segment over time. Labour/Tax, F1 Racing, Tory Politics, Justice, Middle East.

Research Overview & RecSys 2025

Beyond accuracy: Fairness, diversity, serendipity

Fairness

Minimum Exposure Guarantees
(Lopes, Alves et al., Expert Syst. 2024)
ILP post-processing ensuring each item group receives fair exposure.

SAGEA (RecSys 2025 poster)
Fairness-preserving group recommendations via sparse autoencoder aggregation.

Diversity & more

Ontology-based diversity
(Kuznetsov, Kordík, 2023)
Improving cold-start diversity.

Conv4Rec: Serendipitous recommendations.

Regionalization (ACM TSAS 2024)
Spatially-constrained collaborative filtering.

Publications – core methods

Linear autoencoders:

- VASP *RecSys 2021*
- ELSA *RecSys 2022*
- Eval. CFAE @ Scale *ACM TORS 2025*
- Sparse ELSA *WWW 2026*

Factorization & theory:

- Uncertainty-Adj. MF *TNNLS 2023*
- Generalization bounds *ICML 2024*
- Conv4Rec *TNNLS 2025*
- Probabilistic modeling *RecSys 2025*

Sequential & language:

- beeFormer *RecSys 2024*
- CompresSAE *RecSys 2025*
- ReALM *RecSys 2025*
- LLM alignment *ACM TIST 2024*

Publications – applications and evaluation

Bandits, NLP & sequential:

- BMAB *RecSys 2021*
- BPoP *WWW 2024*
- SHIELD (search safety) *UMAP 2025*
- Active Rec. (email) *CIKM 2025*

Applications & evaluation:

- IndAE (cold-start AE) *RecSys 2024*
- Offline/Online eval *KDD 2023*
- Telegraph *RecSys 2025*
- Fairness, diversity, ...

25+ publications at RecSys, WWW, ICML, KDD, TNNLS, TORS, TIST

Open-source projects

ELSA

`recombee/ELSA`

Scalable linear autoencoder, PyPI

beeFormer

`recombee/beeFormer`

Sentence Transformers for RecSys

CompresSAE

`recombee/CompresSAE`

Post-hoc embedding compression

RepSys

`cowjen01/repsys`

Interactive RecSys evaluation

`github.com/recombee`

Research Posters

See all →

recombee.com/research

RecSys 2025 in Prague

The 19th ACM Conference on Recommender Systems

22–26 September 2025, Prague

General Chairs:

M. Bielikova, P. Kordík, M. Schedl

1000+ attendees, around 60% from industry

Google, Amazon, Netflix, Spotify, ...

SlidesLive recordings

Recombee

Empowering Discovery Through Research-Driven Personalization

Recombee delivers real-time personalized recommendations and semantic search powered by the latest in AI research — from proprietary transformer models to reinforcement learning and LLMs.

Our RecSys 2025 Contributions

- The Future is Sparse: **Embedding Compression for Scalable Retrieval** in Recommender Systems
- **Probabilistic Modeling, Learnability and Uncertainty Estimation** for Interaction Prediction in Movie Rating Datasets
- **Recurrent Autoregressive Linear Model** for Next-Basket Recommendation (ReALM)
- SAGEA: **Sparse Autoencoder-based Group Embeddings Aggregation** for Fairness-Preserving Group Recommendations
- **Segment-Aware Analytics** for Real-Time Editorial Support in Media Groups: Lessons from The Telegraph

Who We Are

Prague-based AI company serving billions of recommendations monthly. Trusted by global brands including DAZN, The Telegraph, 9GAG, Audiomack, and Crexi.

Research & Innovation

Efficient Retrieval
Sparse embeddings reduce memory cost while preserving quality.

Uncertainty & Learnability
New provable results and abstention strategies.

Next-Basket Recommendation
ReALM combines interpretability & speed.

Fair Group Recommendations
SAGEA balances fairness and accuracy.

Editorial Analytics
Real-time LLM-based pipelines at The Telegraph.

Why Recombee

- **Real-Time & Scalable** (30k+ recs/s, <200 ms latency)
- **Flexible & Customizable** (ReQL filters, boosters, business logic)
- **Research-Driven** (Beeformer, reinforcement learning, LLM-based semantic search)
- **GDPR-Compliant & Enterprise-Ready**

Explore Our Research

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research@recombee.com github.com/recombee
recombee.com/research

RECOMBEE RESEARCH TEAM

Rodrigo Alves, Ph.D.

Head of Research

Vojtěch Vančura, Ph.D.

ML Researcher

Petr Kasalický

ML Researcher

Dan Bohuněk

Applied Scientist

Jaroslav Hradil

Applied Scientist

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