List of topics for the state doctoral examination (P4M3)
– real, complex, and functional analysis

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1. List **A** - general topics

1.1. Theory of distributions.

- distributions - basic properties, localization and support, convolution of distributions; tempered distributions and Fourier transform; Paley-Wiener theorems, Sobolev lemma, fundamental solution to elliptic equation; [25], pages 149–222

examiners: M. Rokyta, J. Spurný

1.2. Advanced spectral theory.

- deeper properties of the group of invertible elements, Lomonosov theorem, applications of Gelfand transform in non-commutative algebras, eigenvalues of normal operators, roots of positive operators, group of invertible operators, Gelfand-Naimark-Segal construction, ergodic theorem, spectral theorem for unbounded normal operators, operator semigroups; [25], pages 267-271, 292-301, 327-341, 368-385

examiners: V. Müller, J. Spurný

1.3. Complex analysis.

- *Harmonic functions*: Cauchy-Riemann equations, Poisson integral, mean value property, boundary behavior of the Poisson integral, representation theorems; [26], pages 257–276
- *Conformal mapping*: preservation of angles, homograph mappings, normal classes, Riemann theorem (without proof), continuity up to the boundary, conformal map of the anulus; [26], pages 307–314, 318–322
- *Roots of analytic functions*: infinite products, Weierstrass factorization theorem, interpolation problem, Jensen formula, Blaschke products; [26], pages 329–344
- *Analytic continuation*: regular and singular points, continuation along curves, monodromy theorem, construction of modular function, Picard theorem; [26], pages 351–364

examiners: O. Kalenda, M. Rokyta

1.4. **Introduction to abstract harmonic analysis.**

- *Locally compact topological groups*: definition of topological group, Haar measure, modular function, convolution on locally compact abelian groups; [18], pages 31–54
- *Unitary representation*: basic concepts (unitary equivalence, irreducible representation, cyclic representation, commutant), Schur's lemma, functions of positive type, Gelfand-Raikov theorem; [18], pages 67–86
- *Harmonic analysis on LCA groups*: dual group and its topology, Fourier transform, Plancherel's theorem, Pontryagin duality theorem, inversion theorem; [18], pages 87–103

examiners: P. Holický, M. Zelený

1.5. **Introduction to approximation theory.**

- *Existence and uniqueness of best approximation*: basic concepts, best approximatio in finite-dimensional, uniformly and strictly convex Banach spaces, metric projections; [12], pages 20–24
- *Approximations of continuous functions*: Lagrange's formula, Chebyshev polynomials, Hermite's interpolation, Bernstein's polynomials, monotone operators, Fejér's theorem, strong uniqueness, Haar's condition, alternation theorem, Markov systems, theorem of de la Vallée Poussin, Freud's theorem; [12], pages 57–83
- *Kolmogorov's theorem and its consequences:* characterization of best approximation on convex sets via functionals, Fenchel's theorem, Kolmogorov's theorem, Rivlin-Shapiro theorem; [14], pages 58–67
- *Spaces of smooth functions:* modulus of continuity, modulus of smoothness, Marchaud's inequality, Lipschitz spaces; [14], pages 46–54
- *Key approximation theorems:* direct and inverse theorems on trigonometric approximation, Jackson's theorem, Stěčkin's theorem, simultaneous approximations, Zamansky's lemma, Czipszer's and Freud's theorems, Szegö's theorem, Bernstein's inequality, Markov's inequality, Timan-Zygmund's theorem, approximation via algebraic polynomials; [14], pages 97–99, 201–211, 219–223

examiners: Z. Mihula, L. Pick, J. Vybíral, L. Zajíček

1.6. **Classical topics from harmonic analysis.**

- Poisson integral, maximal function, solution to Dirichlet problem for Laplace's equation, Bessel and Riesz potential, relation to Sobolev spaces; Hilbert transform, singular integral operators, Calderón-Zygmund kernel, multipliers in Lebesgue spaces, Mihlin-Hörmander theorem [28, chapters 1–3]

examiners: L. Pick, L. Slavíková, P. Honzík

1.7. Hausdorff measure and change of variables.

- Hausdorff measure and dimension, basic properties, isodiametric inequality, densities, Hausdorff measure and elementrary properties of functions, change of variables for lipschitz transformations: area formula, coarea formula; [16], pages 60–119

examiners: D. Campbell, S. Hencl, J. Vybíral, L. Pick

1.8. Spaces of BV functions and smooth approximations.

- covering theorems, functions of bounded variation, structure theorem, approximation and compactness, traces and extensions, coarea formula for BV functions, isoperimetric inequalities, cantorian part, jumps, sets of finite perimeter, reduced boundary, Gauss-Green theorem for sets with finite perimeter, characterization using the measure of boundary density, approximative differentiability, differentiability in L^p and $W^{1,p}$, convex functions, Whitney's extension theorem, approximation with C^1 functions; [16], pages 166–256

examiners: S. Hencl, L. Pick

2. List **B** - special topics

2.1. Introduction to interpolation theory.

- *Classical interpolation theorems:* Riesz-Thorin theorem, Marcinkiewicz theorem; [10], pages 9–22
- *Abstract interpolation theory*: categories and functors, pairs of spaces, interpolation spaces, Aronszajn-Gagliardo theorem, theorem on duals; [10], pages 34–48
- *Real interpolation method:* Peetre's K-functional and J-functional, Holmstedt's formulas, reiteration theorem, theorem on equivalence of methods, density theorem, monotone interpolation spaces, Cwikel's lemma, K-functional for the pair (L^1, L^∞); [7], pages 293–330, 74–79

examiners: Z. Mihula, L. Pick, B. Opic, L. Slavíková, J. Vybíral

2.2. Topological degree.

- construction and uniqueness of the degree, basic properties; Brouwer's theorem, Borsuk's theorem, product formula and Jordan separation theorem; Leray-Schauder degree: definition and basic properties; [13], pages 1–67

examiners: S. Hencl, J. Spurný

2.3. Integral representation on compact convex sets.

- barycenter of measure, ordering of measures, the Choquet representation theorem, abstract boundaries, Choquet and Bauer simplices; [5], pages 1–55, 84–109

examiners: O. Kalenda, J. Spurný

2.4. Theory of C^*-algebras.

- basic properties of C^*-algebras, ordering of C^*-algebras, positive elements, commutative algebras, representations, topologies on von Neumann algebras, approximation theorems; [22], sections 4.1-4.5 (pages 236–285) and sections 5.1-5.3 (pages 304–330)

examiners: J. Hamhalter, O. Kalenda, V. Müller, J. Spurný

2.5. Descriptive set theory.

- hierarchy of Borel sets, separation and reduction theorems, hierarchy of projective sets, analytic and coanalytic sets, analytically and coanalytically complete sets, spaces of trees and linear orderings, regularity properties of analytic and coanalytic sets, first separation principle, injective Borel maps, Novikov's principle of separation, borel sets with compact sections, polish groups and borel measurability, reduction theorems, theorem on boundedness of coanalytic range, universal sets for $\Delta^1_1(X)$, Choquet's capacitability theorem, second principle of separation, maps with countable point preimages; [27], pages pages 115–182, omitting Solovay's example in section 4.5.3, pages 151–153
- Kuratowski and Ryll-Nardzewski selection thereom; [27], pages 189–193
- Borel sets with σ-compact sections; [27], pages pages 219–227
- Uniformization of coanalytic sets; [27], pages pages 236–240

examiners: B. Vejnar, P. Holický, M. Zelený

2.6. Function spaces.

- *Spaces of integrable and differentiable functions*: Orlicz, Morrey, Campanato spaces, BMO and Sobolev spaces; [24], pages 126–166, 209–229, 249–292

examiners: S. Hencl, Z. Mihula, L. Pick, L. Slavíková, J. Vybíral

2.7. Singular integrals.

- maximal function, covering theorems, Fourier transform of singular integrals, L^2 and weak L^1 estimates, interpolation; [28], pages 3–53
- Riesz transform, spherical harmonics; [28], pages 54–80

examiners: S. Hencl, Z. Mihula, L. Pick, L. Slavíková, J. Vybíral

2.8. Littlewood–Payley theory.

- maximal function, covering theorems, Fourier transform of singular integrals, L^2 and weak L^1 estimates, interpolation; [28], pages 3–53
- Littlewood-Paley theory, multipliers; [28], pages 81–115

examiners: S. Hencl, L. Pick, P. Honzík, L. Slavíková

2.9. **Riesz and Bessel potentials.**

- maximal function, covering theorems, Fourier transform of singular integrals, L^2 and weak L^1 estimates, interpolation; [28], pages 3–53
- Riesz and Bessel potentials, equivalence of Sobolev spaces and spaces of Bessel potentials; [28], pages 116–165

examiners: S. Hencl, P. Honzík, L. Slavíková

2.10. **Hardy spaces.**

- real analytic Hardy spaces, definition based od maximal operators, characterization via atomic decomposition, continuity of singular operators, VMO-H_1-BMO duality, Fefferman-Stein inequality; [29], pages 87–177

examiners: S. Hencl, P. Honzík, L. Slavíková

2.11. **Mappings of finite distortion.**

- definition, distributional Jacobian, weakly monotone mappings, continuity, Lusin (N) and (N^{-1}) conditions, counterexamples; [21], pages 14–39, 63–79

examiners: D. Campbell, S. Hencl

2.12. **Isoperimetric inequality.**

- formulation of the isoperimetric problem, two-dimensional proofs, Federer-Fleming theorem, Gromov's proof, Gagliardo-Nirenberg inequality with an exact constant, Minkowski area, Brunn-Minkowski inequality, isoperimetric inequality for the perimeter; [11], pages 1–100

examiners: S. Hencl, D. Pražák

2.13. **Differentiability of convex functions.**

- *Convex functions*: Gâteaux and Fréchet differentiability, subdifferential, Bishop-Phelps theorem on density of norm attaining functionals, existence of nowhere Fréchet differentiable norm on separable space with nonseparable dual; [8], pages 83–89
- *Null sets*: basic properties of Haar-null sets, difference of Haar non-null sets contains a neighbourhood of neutral element, positive cone in ℓ_p space is Haar-null, Gaussian measures in Banach spaces, Gauss and Aronszajn null sets; [8], pages 125–130, 135–142
- *Points of nondifferentiability*: convex or lipschitz function on a separable Banach space is Gâteaux differentiable up to an Aronszajn null set; [8], pages 153–155

examiners: P. Holický, O. Kalenda, J. Tišer, L. Zajíček

2.14. **Approximations and qualitative aspects of (non)compactness.**

- Finite-dimensional subspaces, approximations, Banach-Mazur distance, Auerbach lemma, principle of local reflexivity; [32], pages 69–79
- Haar functions, Schauder basis, unconditional basis in L^p, Rademacher functions, Khinchin and Kahane-Khinchin inequalities; [4], pages 137–150
- Entropy numbers, Carlo-Triebel inequality, approximation numbers, Weyl inequality, Lorentz sequence spaces; axiomatic definition of s-numbers; examples and relations of the above concepts; measure of noncompactness; generalized Carlo-Triebel inequality; [15], pages 43–77
- Compactness of Sobolev embeddings on bounded domains and on (non)compactness on unbounded domains, necessary condition and quasi-bounded domains, sufficient condition of compactness, characterisation using capacities; [3], pages 167–182
- Compactness of Sobolev embeddings on bounded domains and on (non)compactness on unbounded domains, necessary condition and quasi-bounded domains, sufficient condition of compactness, characterisation using capacities; [3], pages 167–182
- Approximation property of L^p spaces, relation of measure of noncompactness, regularity of boundary and Poincaré inequality; estimates of measure of noncompactness of Sobolev embeddings $W_0^{1,p} \hookrightarrow L^p$ in unbounded domains, asymptotic behavior of approximation numbers for subcritical Sobolev embeddings, and of critical Sobolev embeddings into exponential spaces; [15], pages 276–302

examiners: J. Lang, Z. Mihula, V. Musil, L. Pick, J. Vybíral

2.15. **Structure of separable Banach spaces.**

- theory of Schauder bases, shrinking, boudedly complete, unconditional, duality, bases in classical spaces, Markushevich bases, the structure of spaces l_p, c_0, universality of $C[0,1]$, the Sobczyk theorem on c_0, c_0 is not complementary in ℓ_∞, the injectivity property, Rosenthal theorem on ℓ_1; [17], pages 179–257

examiners: P. Hájek, O. Kalenda, O. Kurka, J. Spurný

2.16. **Spaces of Lipschitz functions and Arens–Eells spaces.**

- *Basic properties of Lipschitz functions:* Basic properties of Lipschitz functions, Rademacher theorem, spaces $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}(M)$, relations between spaces of Lipschitz functions on bounded and unbounded spaces ([31, Sections 1.5–1.6 and 2.1–2.3]).
- *elementary properties of Lipschitz free spaces:* construction of Arens–Eells space, universality property and connection to the space of Lipschitz functions ([31, Sections 3.1–3.3]).
- *Advanced material:* unicity of the predual of $\mathrm{Lip}_0(M)$, w^* extremal points of $\mathcal{F}(M)$, theorem on the structure of isometries for $\mathcal{F}(M)$, where M is a concave metric space, Godefroy–Kalton's result on isometric embedding and extension of Lipchitz maps ([31, Sections 3.4–3.8 and 5.1–5.2]).

examiners: M. Doucha, P. Hájek, M. Johanis, O. Kalenda

2.17. **Function Spaces and Potential Theory.**

- *Capacities and nonlinear potentials:* Sobolev spaces, Bessel potentials, L^p-capacities, the minimax theorem, removability of singularities ([1, Chapters 1,2, pp. 1–48]).
- *Estimates for Bessel and Riesz potentials:* pointwise and integral estimates, sharp exponential estimate, operations on potentials, one-sided approximation, operations on potentials with fractional index, potentials and maximal functions ([1, Chapter 3, pp. 53–81]).
- *Trace and embedding theorems:* a capacitary strong-type inequality, embeddings of potentials, compactness of the embedding, a space of quasicontinuous functions, a capacitary strong-type inequality ([1, Chapter 7, pp. 187–213]).

examiners: Z. Mihula, L. Pick, L. Slavíková

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