$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

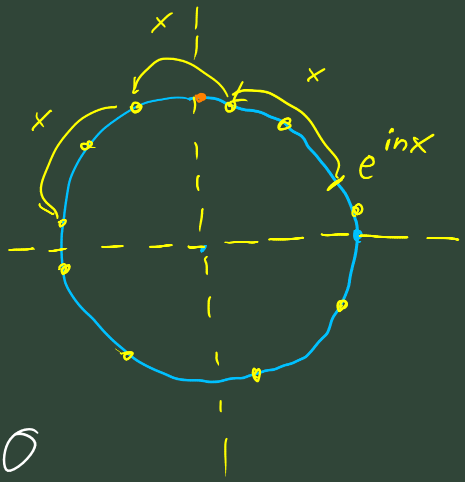
($\cos nx$, $\sin nx$ pro $x \neq 2k\pi$ mají omezené část. součty)

Tyto řady konvergují (plyne z Dirichletova kritéria) pro $x \neq 2k\pi$, $k \in \mathbb{Z}$

konvergují absolutně? (pro $x \neq 2k\pi$)

odpověď: NE

proč?



$$\sum_{n=1}^{\infty} \frac{|\cos nx|}{n} = +\infty$$

$$\min\{|\cos nx|, |\cos (n+1)x|\} \geq \varepsilon > 0$$

Jinak se dá dokázat trikem:

$$\sum_{n=1}^{\infty} \frac{|\cos nx|}{n} \geq \sum_{n=1}^{\infty} \frac{\cos^2 nx}{n} = \sum_{n=1}^{\infty} \frac{1+\cos 2nx}{2n} =$$

$$= \underbrace{\sum_{n=1}^{+\infty} \frac{1}{2n}}_{+\infty} + \underbrace{\sum_{n=1}^{\infty} \frac{\cos 2nx}{2n}}_{\text{pro } x \neq k\pi \text{ je konvergentní}} = +\infty, \quad \text{pro } x \neq k\pi$$

$$\frac{|\cos nx|}{n} \overset{\leq}{\not\geq} \frac{1}{n}$$

Definujme

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad z \in \mathbb{C}, \quad |z| < 1$$

poloměr konvergence ?

$$a_n = \frac{1}{n}, \qquad \limsup_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{1}{\sqrt[n]{n}} = 1$$

$$\Rightarrow \qquad R = \frac{1}{1} = 1$$

$$f'(z) = \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z}, \quad |z| < 1$$

$$f(z) = -\log(1-z), \quad |z| < 1$$

Nechť $x \neq 2k\pi$, potom

$$f(e^{ix}) = -\log(1-e^{ix}) \overset{\text{Abel}}{=} \sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$

$\log z =$
$= \log(r e^{i\theta}) =$
$= \ln r + \log e^{i\theta}$
$= \ln r + i\theta$
$= \ln|z| + i \arg z$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\operatorname{Re}\{\log(1-e^{ix})\}$$
$$= -\ln|1-e^{ix}|$$

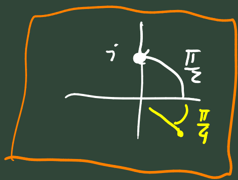
$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\operatorname{Im}\{\log(1-e^{ix})\}$$
$$= -\arg(1-e^{ix})$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln|1-e^{ix}|$$

$$x \neq 2k\pi$$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\arg(1-e^{ix}),$$

- speciálně zvolím $x = \pi \longrightarrow 1 - e^{i\pi} = 2$



$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{\cos \pi n}{n} = -\ln 2$$

- zvolím $x = \frac{\pi}{2} \longrightarrow 1 - e^{i\frac{\pi}{2}} = 1 - i$

$$-\arg(1-i) = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

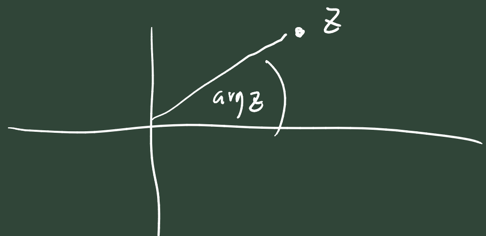
$$\Rightarrow \quad \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{2}n\right)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$$

Def.:

$\arg z$ je číslo $\arg z \in (-\pi, \pi]$ takové, že

$$z = |z|\, e^{i \arg z}, \qquad z \in \mathbb{C} \setminus \{0\}$$



M. Bulíček, část 2

$$\sum_{n=1}^{\infty} \overbrace{\sin\left(\frac{n^2}{n+1}\right)}^{\text{omezené, bude oscilovat} \approx \sin n} \overbrace{\left(1-\cos\left(\frac{1}{n}\right)\right)^p}^{\sim \frac{1}{2n^2}} \arctan\left(n+\frac{1}{n^{1000}}\right)^{\nearrow \frac{\pi}{2}}$$

Vyšetřete konvergenci (absolutní i neabsolutní), kde $p \in \mathbb{R}$ je parametr.

1) <u>nutná podmínka konvergence</u>

$$b_n = \left(1-\cos\frac{1}{n}\right)^p = \left(\frac{1}{2n^2} + O\left(\frac{1}{n^4}\right)\right)^p =$$

$$= \left(\frac{1}{2n^2}\right)^p \left(1 + o\left(\frac{1}{n^2}\right)\right)^p = \left(\frac{1}{2n^2}\right)^p \left(1 + o\left(\frac{1}{n^2}\right)\right)$$

odtud plyne: $b_n \longrightarrow 0$ právě když $p > 0$

$\rightarrow$ nutná pod. konvergence je splněná právě když $p > 0$

($\sin\left(\frac{n^2}{n+1}\right)$ osciluje, $\arctan\left(n+\frac{1}{n^{1000}}\right) \longrightarrow \frac{\pi}{2}$)

Řada D pro $p \leq 0$

2) absolutní konv.

$$\left|\sin\frac{n^2}{n+1}\right| \leq 1$$

$$\left|\arctan\left(n+\frac{1}{n^{1000}}\right)\right| \leq \frac{\pi}{2}$$

Takže

$$\sum 1 \cdot 1 \leq \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n^2}\right)^p \left(1 + o\left(\frac{1}{n^2}\right)\right) < +\infty, \text{ pro } p > \frac{1}{2}$$

Závěr: Řada AK pro $p \in \left(\frac{1}{2}, +\infty\right)$

3) Nechť $p \in \left(0, \frac{1}{2}\right]$

$\sum_{n=1}^{\infty} a_n b_n c_n$ kde

$a_n = \sin\left(\frac{n^2}{n+1}\right)$

$b_n = \left(\frac{1}{2n^2}\right)^p \left(1 + O\left(\frac{1}{n^2}\right)\right)$

$c_n = \arctan\left(n + \frac{1}{n^{1000}}\right)$

$$a_n = \sin\left(\frac{n^2 - 1 + 1}{n+1}\right) = \sin\left(n + 1 + \frac{1}{n+1}\right) =$$

$$= \sin(n+1) + \cos(n+1) \cdot \frac{1}{n+1} - \frac{1}{2}\sin(n+1) \cdot \left(\frac{1}{n+1}\right)^2 + \cdots$$

$$= \sin(n+1) + O\left(\frac{1}{n}\right)$$

$$c_n = \arctan\left(n + n^{-1000}\right) = \arctan n + \frac{1}{1+n^2} \cdot n^{-1000} - \frac{1}{2}\frac{2n}{(1+n^2)^2} n^{-2000} + \cdots$$

$$= \arctan n + O\left(\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty} a_n b_n c_n = \sum_{n=1}^{\infty} \left[\sin(n+1) + O\left(\frac{1}{n}\right)\right]\left[\left(\frac{1}{2n^2}\right)^p + O\left(\frac{1}{n^{4p}}\right)\right]\left[\arctan n + O\left(\frac{1}{n}\right)\right]$$

$$= \sum_{n=1}^{\infty} \left[\sin(n+1)\left[\left(\frac{1}{2n^2}\right)^p + O\left(\frac{1}{n^{4p}}\right)\right] \arctan n + \underbrace{O\left(\frac{1}{n^{1+2p}}\right)}_{AK}\right]$$

stačí se dále zabývat jenom řadou

$$\sum_{n=1}^{\infty} \sin(n+1) \left(1-\cos\frac{1}{n}\right)^p \arctan n$$

$$b_n = \left(1-\cos\frac{1}{n}\right)^p \searrow 0 \quad , \text{ pro } \quad n \to \infty$$

$\Rightarrow$ $\sum_{n=1}^{\infty} \sin(n+1)\, b_n$ konv. (dle Dirichletova kritéria)

$\Rightarrow$ $\sum_{n=1}^{\infty} \overbrace{\sin(n+1)\, b_n}^{a_n} \arctan n$ konv. (dle Abelova kritéria)

Závěr: konv. pro $p \in \left(0, \frac{1}{2}\right]$

4) Nechť $p \in \left(0, \frac{1}{2}\right]$

$$\sum_{n=1}^{\infty} |\sin(n+1)\, b_n \arctan n| \geq \sum_{n=1}^{\infty} \overbrace{\sin^2(n+1)}^{\frac{1-\cos(2(n+1))}{2}} |b_n \arctan n| =$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} |\overset{\sim \frac{1}{(2n)^p}}{b_n} \arctan n|}_{=+\infty} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} \cos(2n+2)\, b_n \arctan n}_{\text{Konv. (viz 3)}} = +\infty$$

Úplný def. závěr:

p	$(-\infty, 0]$	$\left(0, \frac{1}{2}\right]$	$\left(\frac{1}{2}, +\infty\right)$
	D	NAK	AK

Přestávka: pokračování v 12:50

5) $$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad , \; z \in \mathbb{C}$$

$p \in \mathbb{R}$ parametr

Vyšetřete:
- poloměr konvergence
- chování na kružnici

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \frac{1}{\left(\lim\limits_{n\to\infty} \sqrt[n]{n}\right)^p} = 1$$

$\rightarrow \quad R = 1$

$\rightarrow \quad \sum_{n=1}^{\infty} \frac{z^n}{n^p}$ AK pro $|z| < 1$

Jak je to s $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$ pro $|z| = 1$?

- $p > 1$: srovnávací krit $\left(\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \left| \frac{z^n}{n^p} \right| \right)$

  $\rightarrow$ AK

- $p \leq 0$: není splněna nutná pod. konv

  $\rightarrow$ D

- $p \in (0, 1]$:

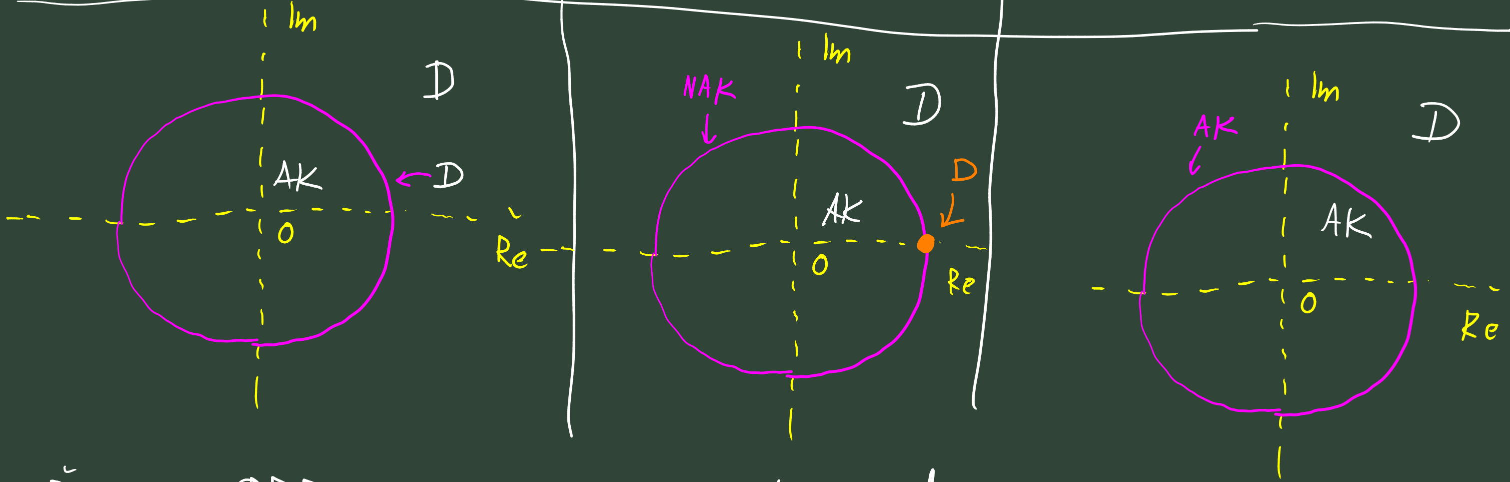
$z = 1 \ \dots \ D$

$z \neq 1 \longrightarrow z = e^{i\theta}$

$\theta \in (0, 2\pi)$

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n^p} \ \dots \ \text{NAK}$$

$p \in [-\infty, 0]$	$p \in (0, 1]$	$p \in (1, +\infty)$



Řešení ODR pomocí mocninných řad

$$y'' - 2xy' + y = 0$$

hledáme řešení ve tvaru mocninné řady

Ansatz: $y(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, kde $a_n \in \mathbb{R}$

$$y' = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}$$

$$y'' = \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}$$

$$\sum_{n=2 \,\downarrow -2}^{\infty} a_n \frac{x^{n-2 \,\uparrow +2}}{(n-2)!} - \sum_{n=1}^{\infty} 2a_n \frac{x^n}{(n-1)!} + \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!} - \sum_{n=1}^{\infty} 2na_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = 0$$

$$\sum_{n=1}^{\infty} \left(a_{n+2} - 2na_n + a_n\right) \frac{x^n}{n!} + a_{n+2} + a_n = 0$$

mocniné řady jsou jednoznačné určené svými koeficienty

$$a_{n+2} - 2na_n + a_n = 0, \quad n = 0, 1, 2, \dots, \infty$$

$$\boxed{a_{n+2} = (2n-1)\, a_n}, \quad n = 0, 1, 2, \dots, \infty$$

$$\rightarrow \quad a_0, a_1 \in \mathbb{R} \quad \& \quad a_{2n+1} = \big(2(2n-1) - 1\big) a_{2n-1}$$

$$= (4n-3)\, a_{2n-1}$$

$7 = 2\cdot 3 + 1$

$4\cdot 3 - 3$

$$a_3 = 1\cdot a_1 \qquad\qquad a_2 = -a_0$$

$$a_5 = 5\cdot 1\cdot a_1 \qquad\qquad a_4 = -3\cdot a_0$$

$$a_7 = 9\cdot 1\cdot a_1 \qquad\qquad a_6 = -7\cdot 3\cdot a_0$$

$$\vdots \qquad\qquad \vdots$$

$6 = 2\cdot 3$

$7 = 4\cdot 3 - 5$

$$\boxed{\begin{aligned} a_{2n+1} &= \left(\prod_{k=1}^{n} [4k-3]\right) a_1 \\ a_{2n} &= \left(\prod_{k=1}^{n} [4k-5]\right) a_0 \end{aligned}}$$

$$y(x) = \underbrace{a_0 \sum_{k \text{ sudé}} \left(\prod \cdots\right) \frac{x^k}{k!}}_{u_1} + \underbrace{a_1 \sum_{k \text{ liché}} \left(\prod \cdots\right) \frac{x^k}{k!}}_{u_2}$$

$$u_1 = \sum_{n=0}^{\infty} a_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} 4k-3\right) \frac{x^{2n+1}}{(2n+1)!}$$

poloměr konv. ? $\quad \limsup_{n\to\infty} \sqrt[n]{|A_n|}, \quad A_n = \dfrac{\prod_{k=1}^{n}(4k-3)}{(2n+1)!}$

$$\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} \frac{4n+1}{(2n+3)(2n+1)} = 0, \quad R = +\infty \longrightarrow u_1, u_2 \text{ jsou definovány na } \mathbb{R}$$