(Přestávka : pokračování ve 12:40)

$$f = \left(e^{x} \cdot \cos Bx\right) g_1 + e^{3x} g_2$$

Řady s nezápornými členy

$$Ř = \sum_{n=1}^{\infty} a_n \quad , \qquad a_n \in [0,+\infty) , \quad n = 1,2,3,\ldots$$

$$Ř = \lim_{M\to\infty} \sum_{n=1}^{M} a_n \in [0,+\infty]$$

④ Sečtěte

$$\sum_{n=1}^{\infty} \frac{1}{n\underset{3}{(n+1)}} = ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{M\to\infty} \sum_{n=1}^{M} \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$= \lim_{M\to\infty} \left(\sum_{n=1}^{M} \frac{1}{n} - \sum_{n=1}^{M} \frac{1}{n+1} \right) = \lim_{M\to\infty} \left(\sum_{n=1}^{M} \frac{1}{n} - \sum_{n=2}^{M+1} \frac{1}{n} \right) =$$

$$= \lim_{M\to\infty} \left(\frac{1}{1} - \underbrace{\frac{1}{M+1}}_{\searrow 0} \right) = \underline{\underline{1}}$$

Dále se budeme zabývat jen konvergencí (nepodaří se nám řadu sečíst)

Srovnávací kritérium

$0 \le a_n \le b_n$, $n=1,2,\dots$

- $\sum\limits_{n=1}^{\infty} b_n < +\infty \Rightarrow \sum\limits_{n=1}^{\infty} a_n < +\infty$
- $\sum\limits_{n=1}^{\infty} a_n = +\infty \Rightarrow \sum\limits_{n=1}^{\infty} b_n = +\infty$

Limitní srovnávací kritérium

$0 \le a_n$, $0 \le b_n$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = r \in (0,+\infty)$

Potom: $\sum\limits_{n=1}^{\infty} a_n$ K $\iff$ $\sum\limits_{n=1}^{\infty} b_n$ K

Je dobré si pamatovat:

- $\sum\limits_{n=1}^{\infty} q^n = \overset{\frac{q}{1-q}}{K} \iff |q| < 1$
- $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ K $\iff \alpha > 1$ ← je stejné jako u $\int_1^{+\infty} \frac{1}{x^a}\,dx$

Nutná podmínka konvergence

$$a_n \not\to 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \ \text{ nekonverguje}$$

6) $$\sum_{n=1}^{\infty} \tan\left(\frac{\pi}{4n}\right)$$

$\Big(\frac{\pi}{4n} \in \left(0, \frac{\pi}{4}\right]$

$\to$ dobře definováno

$\to$ $\tan\left(\frac{\pi}{4n}\right) \geq 0 \Big)$

- $$\frac{\pi}{4n} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \quad \tan\left(\frac{\pi}{4n}\right) \xrightarrow{n\to\infty} 0 \quad \Rightarrow$$ je splněna nutná podmínka

- $$\tan\left(\frac{\pi}{4n}\right) \sim \frac{\pi}{4n}, \quad n \to \infty$$

$$\left(\frac{\tan x}{x} \to 1, \quad x = \frac{\pi}{4n} \right)$$

Víme $$\sum \frac{\pi}{4n} = \frac{\pi}{4} \sum \frac{1}{n} = +\infty$$

$\Rightarrow$ $\sum_{n=1}^{\infty} \tan \frac{\pi}{4n}$ diverguje dle limitního srovnávacího kritéria

13) $$\sum_{n=1}^{\infty} \left(n^{\frac{1}{n^2+1}} - 1 \right)$$

$$a_n = n^{\frac{1}{n^2+1}} - 1 = \exp\left(\overset{\nearrow 0}{\frac{\ln n}{n^2+1}} \right) - 1$$

Použiju limitu $$\frac{e^x - 1}{x} \to 1, \quad x \to 0$$

$$\Rightarrow \qquad a_n \sim b_n = \frac{\ln n}{n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1} = ?$$

Srovnám s $\quad c_n = \dfrac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{\ln n}{n^2+1} \cdot n^{3/2} = \frac{\ln n}{\sqrt{n} + n^{-3/2}} \xrightarrow{n\to\infty} 0$$

$$\Rightarrow \qquad b_n \leq c_n \quad \text{pro} \quad \text{velká } n$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < +\infty \qquad \left(\tfrac{3}{2} > 1\right)$$

srovnávací kritérium

$$\Rightarrow \qquad \sum_{n=1}^{\infty} b_n < +\infty$$

lim. srovnávací kritérium

$$\Rightarrow \qquad \xi = \underbrace{\sum_{n=1}^{\infty} a_n < +\infty}$$

⑦

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right) =$$

$$\sim \frac{1}{n^{3/2}}, \quad \sum \frac{1}{n^{3/2}} K$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \frac{(n+1)-(n-1)}{\sqrt{n+1}+\sqrt{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{n} \frac{2}{\sqrt{n+1}+\sqrt{n-1}}\right) \ldots K$$

(lim. srov. kritérium)

A) podílové kritérium

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

- $q > 1 \implies$ řada $\sum a_n$ diverguje
- $q < 1 \implies$ řada $\sum a_n$ konverguje

($q = 1$, kritérium nedává žádnou informaci)

B) odmocninové kritérium

$$q = \limsup_{n\to\infty} \sqrt[n]{a_n}$$

- $q > 1 \implies$ řada $\sum a_n$ diverguje
- $q < 1 \implies$ řada $\sum a_n$ konverguje

($q = 1$, kritérium nedává žádnou informaci)

Př. ⑭

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{1 \cdot 5 \cdot 9 \cdots (4n-3)} \right)}_{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{3n+4}{4n+1} \xrightarrow{n\to\infty} \frac{3}{4} = q < 1 \implies \text{řada } K$$

⑯

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(\frac{\pi}{3}+\frac{1}{n}\right)^n}$$

$$q = \limsup_{n\to\infty} \sqrt[n]{\frac{n^2}{\left(\frac{\pi}{3}+\frac{1}{n}\right)^n}} \overset{?}{=} \lim_{n\to\infty} \frac{\Big(\overset{\to 1}{\boxed{\sqrt[n]{n}}}\Big)^2}{\frac{\pi}{3}+\underset{\to 0}{\boxed{\frac{1}{n}}}} =$$

$$= \frac{3}{\pi} < 1 \quad \dots\dots \quad K$$ dle odmocninového kritéria

⑩

$$\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$$

$$\sqrt[n]{a_n} = \frac{n^{1+\frac{1}{n^2}}}{n+\frac{1}{n}} = \frac{n\cdot\sqrt[n^2]{n}}{n+\frac{1}{n}} = \frac{\overset{\to 1}{\boxed{\sqrt[n^2]{n}}}}{1+\underset{\to 0}{\boxed{\frac{1}{n^2}}}}$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_n} = 1 \Rightarrow$$ kriterium nedává žádnou informaci

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+\frac{1}{n}}\right)^n \cdot \underbrace{\sqrt[n]{n}}_{\to 1} =$$

$$= \lim_{n\to\infty} \exp\left(n \ln\left(\frac{1}{1+\frac{1}{n^2}}\right)\right)$$

$$\overset{\cancel{?}}{=} \exp\left(\lim_{n\to\infty} n \ln\left(\underbrace{\frac{1}{1+\frac{1}{n^2}}}_{\downarrow\, 1}\right)\right)$$

použiju

$\ln x \sim x-1$

pro $x \to 1$

$$= \exp\left(\lim_{n\to\infty} n\left(\frac{1}{1+\frac{1}{n^2}}-1\right)\right)$$

$$= \exp\left(\lim_{n\to\infty} n\left(\frac{-\frac{1}{n^2}}{1+\frac{1}{n^2}}\right)\right)$$

$$= \exp\left(-\lim_{n\to\infty} \underset{\overset{\downarrow}{0}}{\frac{1}{n}} \cdot \underset{\overset{\downarrow}{1}}{\frac{1}{1+\frac{1}{n^2}}}\right) = \exp 0 = 1$$

Není splněna nutná podmínka konvergence !

$\rightarrow \quad \sum a_n \quad D$

(chová se jako: $1+1+1+1+1+\cdot\cdot$)