První DÚ

i) $\int_0^1 x^p\,dx$, ii) $\int_1^{+\infty} x^p\,dx$, iii) $\int_0^{+\infty} x^p\,dx$, $p \in \mathbb{R}$

i) K pro $p > -1$

ii) K pro $p < -1$

iii) D vždycky

$$\int_{\substack{|x|<1 \\ x \in \mathbb{R}^d}} |x|^p\,dx \;\dots\; K \text{ pro } p > -d$$

Speciálně

$$\int_M \frac{d\vec{r}}{r} \;\dots\; K$$ pro omezenou množinu $M \subset \mathbb{R}^3$

$$\int_0^{+\infty}\left(1-e^{-1/x}\right)dx \quad \dots \quad K/D$$

 :

i) $f(0^+) = 1$

ii) $f(x) \sim \frac{1}{x}, \ x \to +\infty$

$$f(x) = 1 - e^{-1/x}$$

$$\int_{10\,000}^{+\infty} \frac{dx}{x} = +\infty \quad \xrightarrow{\text{LSK}} \quad \int_{10\,000}^{+\infty} f(x)\,dx = +\infty$$

(z ii) $\rightarrow$ $f(x) > 0.99 \cdot \frac{1}{x}$ pro velká x

$\rightarrow \int_{10\,000}^{+\infty} f(x)\,dx > 0.99 \int_{10\,000}^{+\infty} \frac{dx}{x} = +\infty$)

$$1 - e^{-1/x} = 1 - \left(1 - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \dots\right), \ \forall x > 0$$

$$= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \dots, \ \forall x > 0$$

$$\geq \frac{1}{x} - \frac{1}{2x^2}, \ \forall x > 1$$

$$\int_{10\,000}^{+\infty}\left(1 - e^{-1/x}\right)dx \geq \int_{10\,001}^{+\infty}\left(\frac{1}{x} - \frac{1}{2x^2}\right)dx = +\infty - \alpha^{\in \mathbb{R}} = +\infty$$

$f(x) = x^{-a}, \quad x \in (1, +\infty), \quad a > 0$

$$S = 2\pi \int_1^{+\infty} |f(x)| \sqrt{1 + (f'(x))^2} \, dx$$

$$= 2\pi \int_1^{+\infty} x^{-a} \sqrt{1 + \underset{\sim\sim\sim}{a^2 x^{-2a-2}}} \, dx$$

$$f(x) = x^{-a} \sqrt{1 + a^2 x^{-2a-2}}$$

platí:

$$\lim_{x \to \infty} \frac{f(x)}{x^{-a}} = 1 \iff f(x) \sim x^{-a}$$

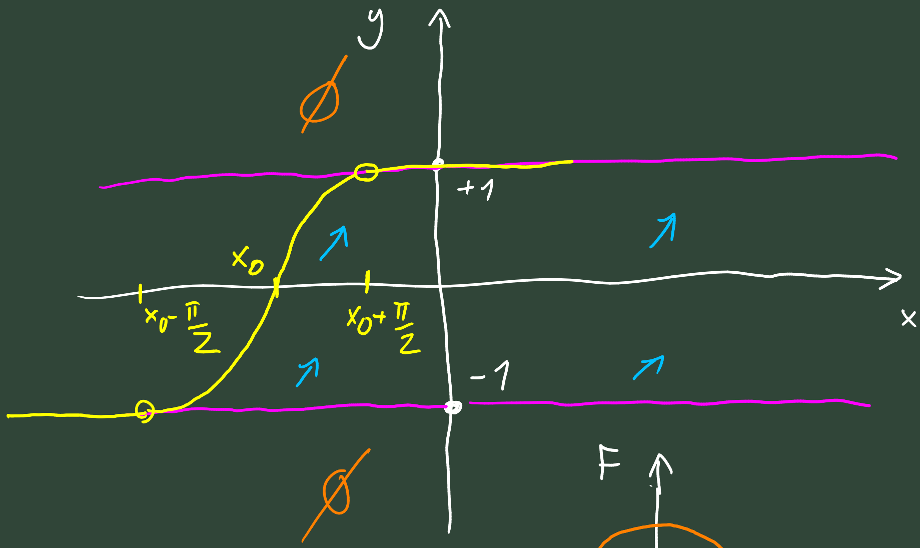
LSK:

$$\int_1^{+\infty} f(x)\,dx \ K \iff \int_1^{+\infty} \frac{dx}{x^a} \ K$$

$$\iff a > 1$$

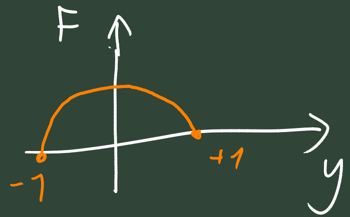
$$-\|- \ D \iff \boxed{a \le 1}$$

5) $y' = \underbrace{\sqrt{1-y^2}}_{F(x,y)}$



$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx$$

$$\arcsin y = \underbrace{x - x_0}_{x \in R(\arcsin) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$



$$y = \sin(x - x_0), \quad x - x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \dots$$

není max
lze prodloužit ±1

Závěr:

- $y \equiv \pm 1$

- $$y(x) = \begin{cases} \sin(x-x_0), & x \in \left(x_0 - \frac{\pi}{2}, x_0 + \frac{\pi}{2}\right) \\ -1, & x \leq x_0 - \frac{\pi}{2} \\ +1, & x \geq x_0 + \frac{\pi}{2} \end{cases}$$

Homogenní rovnice

$$\frac{dy}{dx} = y' = F\left(\frac{y}{x}\right) \neq f(x)\,g(y)$$

obecněji:

$$y' = F\left(\frac{y+b}{x+a}\right), \quad a,b \in \mathbb{R}$$

$$\boxed{u = \frac{y+b}{x+a}} \rightarrow \quad y+b = (x+a)u$$

$$y' = (x+a)u' + 1\cdot u$$

$$(x+a)u' = F(u) - u$$

$$u' = \frac{F(u)-u}{x+a}$$

převedo na předchozí případ ... vyřeším separací

$$\int \frac{du}{F(u)-u} = \int \frac{dx}{x+a}$$

$$y' = \frac{x-y+1}{x+y-3} \overset{?}{=} F\left(\frac{y+b}{x+a}\right)$$

$$= \frac{(x-1)-(y-2)}{(x-1)+(y-2)} = \frac{1-\frac{y-2}{x-1}}{1+\frac{y-2}{x-1}}$$

sub.

$$u := \frac{y-2}{x-1}$$

$$(x-1)u = y-2 \quad \Big| \frac{d}{dx}$$

$$u + (x-1)u' = y'$$

$$(x-1)u' + u = \frac{1-u}{1+u}$$

$$(x-1)u' = \frac{1-2u-u^2}{1+u}$$

$$\int \frac{(1+u)}{1-2u-u^2}\,du = \int \frac{dx}{x-1}$$

$$LS = -\frac{1}{2}\int \frac{\frac{d}{du}(u^2+2u-1)}{u^2+2u-1}\,du \overset{c}{=} -\frac{1}{2}\ln|u^2+2u-1|$$

$$PS \overset{c}{=} \ln|x-1|$$

$$u^2+2u-1 = \frac{C}{(x-1)^2}$$

$$\rightsquigarrow \quad y(x) = ?$$

Přestávka

pokrač: 12:40

Metoda integračního faktoru

Už umíme: $y' = f(x)g(y)$

$y' = f\left(\frac{y+b}{x+a}\right)$

lineární případ

Jak řešit: $y' + f(x)\,y = g(x)$ (1)

i) Najdu PF k f: $F'(x) = f(x)$

ii) Definuji $\gamma(x) := \exp(F(x))$ ← tomu se říká integrační faktor

iii) vynásobí rovnici (1) integračním faktorem:

$$\underbrace{e^{F(x)}y' + f(x)e^{F(x)}y} = g(x)e^{F(x)}$$

$$\left(e^{F(x)}y\right)' = g(x)e^{F(x)}$$

iv) zintegruju $\int_{x_0}^{x} d\xi$ & vydělím $\gamma = e^{F(x)}$

$$\int_{x_0}^{x}\left(e^{F(\xi)}y(\xi)\right)'d\xi = \int_{x_0}^{x} g(\xi)\,e^{F(\xi)}\,d\xi$$

$$\boxed{y(x) = Ce^{-F(x)} + e^{-F(x)}\int_{x_0}^{x} g(\xi)\,e^{F(\xi)}\,d\xi}$$

$\boxed{3}$

$$y' + \boxed{2x}\,y = 2x\,e^{-x^2}$$

$$f(x) = 2x$$

$$F(x) = x^2$$

$$\gamma = e^{F(x)} = e^{x^2}$$

$$y' e^{x^2} + 2x e^{x^2} y = 2x$$

$$\frac{d}{dx}\left(y e^{x^2} \right) = 2x$$

$$y e^{x^2} = C + x^2$$

$$\underline{y(x) = \left(C + x^2\right) e^{-x^2}}, \quad x \in \mathbb{R}$$

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$$y' - 2\frac{y}{x} = x^3$$

$$\left[f(x) = -\frac{2}{x}, \quad F(x) = -2\ln|x|, \quad \gamma = e^{F(x)} = \frac{1}{x^2} \right]$$

$$\left(\frac{y}{x^2}\right)' = \frac{y'}{x^2} - \frac{2}{x^3}y = x \longrightarrow \frac{y}{x^2} = C + \frac{x^2}{2}$$

$$\rightarrow \underline{y(x) = x^2\left(C + \frac{x^2}{2}\right)}, \quad x \in (0, +\infty) \text{ nebo } (-\infty, 0)$$

$\boxed{5}$ $(xy)' = xy' + y = \ln x + 1$

Není ve tvaru $y' = F(x,y)$ [explicitní]

ale v tomto případě $\longleftrightarrow$ $\boxed{y' + \frac{y}{x} = \frac{\ln x + 1}{x}}$

Obecně lze řešit

$$G(x, y, y') = 0$$ [implicitní]

$$y(x) = \ln x + \frac{C}{x}, \quad x > 0$$

$$y' = \frac{1}{x} - \frac{C}{x^2} \rightarrow y' + \frac{y}{x} = \frac{1}{x} - \frac{C}{x^2} + \frac{\ln x}{x} + \frac{C}{x^2} = \frac{\ln x + 1}{x}$$

Bernoulliova ODR

$$y' + p(x)\,y(x) = q(x)\,y^{\alpha}(x)$$

p, q jsou fce, $\alpha \in \mathbb{R}$ konst.

Případ $\alpha = 0$: $\quad y' + p(x)\,y = q(x)$

$\alpha = 1$: $\quad y' + [p(x) - q(x)]\,y = 0$

řeším metodou IF

Případ $\alpha \in \mathbb{R} \setminus \{0, 1\}$

i) vydělím $y^{\alpha}(x)$

$$\boxed{y^{-\alpha} y'} + p\,\boxed{y^{1-\alpha}} = q$$

ii) sub. $y \longmapsto z = \boxed{y^{1-\alpha}}$

$$\frac{dz}{dx} = (1-\alpha)\,\boxed{y^{-\alpha}\,y'}$$

$\to$ $$\frac{z'}{1-\alpha} + p z = q$$

$\to$ vyřeším metodou IF $\longrightarrow$ zpětná substituce

10)

$$y' - \frac{y}{x} = \frac{1}{2y} \quad \Big| \cdot y \quad , \quad \alpha = -1$$

$$yy' - \frac{1}{x}\underbrace{\boxed{y^2}}_{=:z} = \frac{1}{2}$$

$z = y^2$

$z' = 2yy'$

$$\longrightarrow \quad \frac{1}{2}z' - \frac{z}{x} = \frac{1}{2}$$

$$z' - \frac{2}{x}z = 1 \quad \Big| \cdot \frac{1}{x^2}$$

$$\left(\frac{z}{x^2}\right)' = \frac{1}{x^2}$$

$$\frac{z}{x^2} = c - \frac{1}{x}$$

$$y^2 = z = cx^2 - x = x(cx - 1)$$

$$\rightarrow \quad y = \pm\sqrt{x(cx-1)}$$

Omezím na $c \neq 0$, $c = \frac{1}{x_0}$

i) $y = \sqrt{x}$

ii) $y = -\sqrt{x}$

iii) $y = \sqrt{x\left(\frac{x}{x_0} - 1\right)}$

iv) $y = -\sqrt{x\left(\frac{x}{x_0} - 1\right)}$

$\rightarrow$ diskuze ....