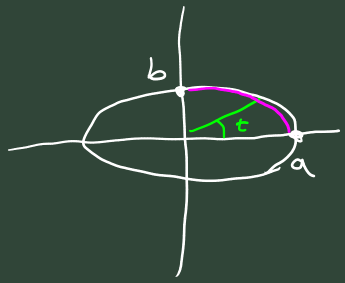
Délka elipsy

$$x(t) = a\cos t,\quad y(t) = b\sin t, \quad t \in [0, 2\pi],$$

$$a > b > 0$$

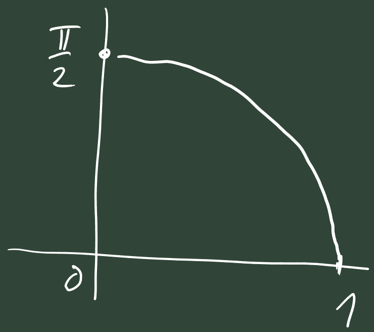


$$L = 4\int_0^{\pi/2} \sqrt{x^2 + y^2}\, dt = 4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt =$$

$\cos^2 + \sin^2 = 1$

$$= 4a \int_0^{\pi/2} \sqrt{1 - \cos^2 t + \frac{b^2}{a^2}\cos^2 t}\, dt = 4a \int_0^{\pi/2} \sqrt{1 + \underbrace{\left(\frac{b^2}{a^2} - 1\right)}_{k^2}\cos^2 t}\, dt$$

$$= 4a \overbrace{\int_0^{\pi/2} \sqrt{1 + k^2\cos^2 t}\, dt}^{E(k)} = 4a\,E(k)$$



$$\int R(x, \sqrt{q})\, dx$$

Diferenciální rovnice

$$y' = F(x,y)$$

$$F : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

interval (otevřený)

Řešením rozumím funkci $y : I \to \mathbb{R}$ splňující

$$y'(x) = F(x, y(x)), \quad \forall x \in I$$

Max. řešení = řešení (y, I) takové, že neexistuje $(\tilde{y}, \tilde{J})$ s vlastností $I \subset \tilde{J}$, $y = \tilde{y}$ na I.

Separace proměnných

Uvažujme F ve tvaru $F(x,y) = g(y)\, f(x)$

$$\frac{dy}{dx} = F(x,y) = g(y) f(x) \quad \Big| \cdot \frac{dx}{g(y)}$$

$$H = \int \frac{dy}{g}$$

$$F = \int f\, dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

$$H(y) = F(x) + C$$

$$y = H^{-1}(F(x) + C)$$

když $y(x) = H^{-1}(F(x)+C)$

$$H(y(x)) = F(x) + C \quad \Big| \frac{d}{dx}$$

$$H'(y(x))\, y'(x) = F'(x)$$

$$\frac{y'(x)}{g(y)} = f(x) \quad \Big| \cdot g(x)$$

$$y'(x) = f(x)\, g(y) = F(x, y(x))$$

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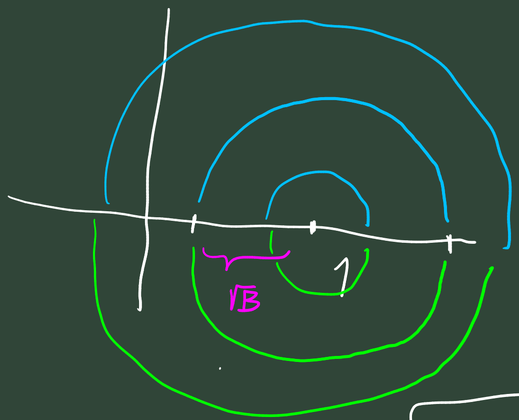
hledání PF: $y' = f(x)$

$\boxed{3}$

$$y' = \frac{1-x}{y}$$

:

* $y(x) \neq 0$
* $y'(1) = 0$



$$\frac{dy}{dx} = \frac{1-x}{y}$$

$$\int y\, dy = \int (1-x)\, dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + \frac{C}{2}$$

$$y = \pm\sqrt{2x - x^2 + C}$$

$$y = \pm\sqrt{B - (1-x)^2}$$

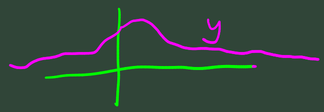
$$x \in (1-\sqrt{B},\ 1+\sqrt{B})$$

Kdy je řešení (y, I) maximální

$\iff$ nelze prodloužit

$\iff$ pro každý krajní bod a intervalu I nastane jedna ze tří možností

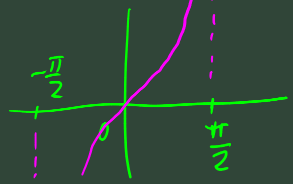
i) $a = \pm\infty$



ii) $\lim\limits_{x \to a^+} (x, y(x)) \notin D(F)$

iii) $\lim\limits_{x \to a^+} y(x) = \pm\infty$

$\frac{d}{dx}(\tan x) = \tan^2 x + 1$

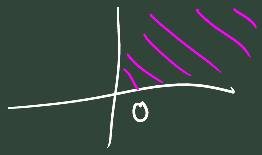


$\boxed{2}$

$$y' = \frac{\sqrt{y}}{\sqrt{x}}$$

:

* $F(x,y)$, $D(F) = (0,+\infty) \times (0,+\infty)$



* $y \equiv 0$ je max. řešení na intervalu $x \in (0,+\infty)$

$$\frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x}} \longrightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{dx}{\sqrt{x}}$$

$$2\sqrt{y} = 2\sqrt{x} + 2C$$

$$\sqrt{y} = \sqrt{x} + C \quad \Big|\,{}^{\wedge}2$$

$$y = (\sqrt{x} + C)^2, \quad x \in (0,+\infty)$$

ZK: $y' = 2(\sqrt{x} + C) \cdot \frac{1}{2\sqrt{x}} \stackrel{?}{=} \frac{\sqrt{y}}{\sqrt{x}} = \frac{1}{\sqrt{x}}\,|\sqrt{x} + C|$

ODR splněna pro $\sqrt{x} + C > 0 \iff x > C^2$, pro $C < 0$

$x \in (0,+\infty)$, pro $C \geq 0$

našli jsme řešení ve tvaru

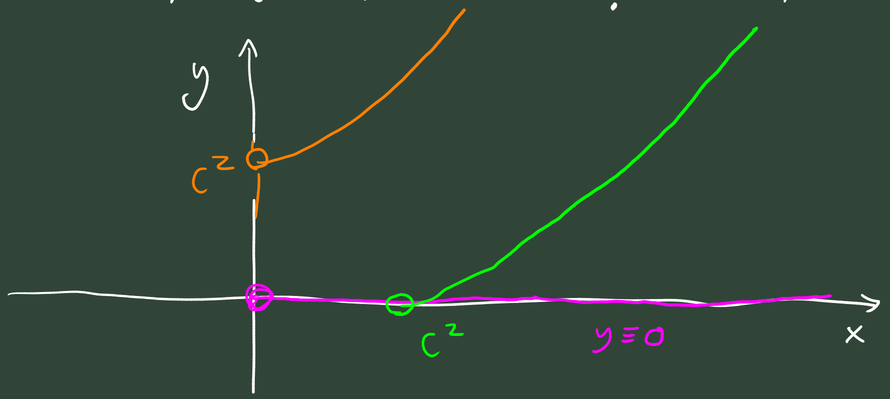
$$\left.\begin{aligned} y &= (\sqrt{x} + C)^2, \quad x \in (0,+\infty) \\ y &= (\sqrt{x} - C)^2, \quad x \in (C^2,+\infty) \end{aligned}\right\} \text{ kde } C \geq 0$$

- $y = (\sqrt{x} + c)^2, \quad x \in (0, +\infty)$
- $y = (\sqrt{x} - c)^2, \quad x \in (c^2, +\infty)$

kde $c \geq 0$

- $y \equiv 0$

Jsou tato řešení maximální ? : Odpověď zní, že nikoliv



řešení
$y = (\sqrt{x} - c)^2$
lze prodloužit jako
$y(x) = 0, \quad x \in (0, c^2)$

$$\lim_{x \to c^2 +} y'(x) = \lim_{x \to c^2 +} (\sqrt{x} - c)\frac{1}{\sqrt{x}} = 0 = \lim_{x \to c^2 -} y'(x)$$

$\boxed{4}$

$$y' = -\frac{e^x}{2y(1+e^x)},$$

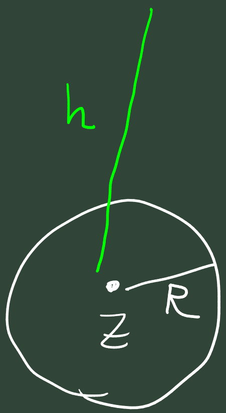
řešení $y = \pm\sqrt{\ln\left(\frac{1+e^{x_0}}{1+e^x}\right)}$
$x \in (-\infty, x_0)$

$\boxed{5}$

$$y' = \sqrt{1-y^2},$$

řešení $y = \sin(x - x_0), x \in (x_0 - \frac{\pi}{2}, x_0 + \frac{\pi}{2})$
(lze rozšířit ± 1 na $\mathbb{R}$)

$\boxed{1.4}$



r ... vzdálenost meteroidu T od Země

$\dot{r} = v$

$v = v(r)$, $v_t = v(R) = ?$

m ... hmotnost T

M ... hmotnost Z

G ... gravitační konstanta

$$F = G\frac{mM}{r^2},$$

$$\dot{v} = \ddot{r} = -\frac{F}{m}$$

$$\boxed{\ddot{r} = -\frac{GM}{r^2}} \quad \Big| \cdot \dot{r}$$

$$\ddot{r}\dot{r} = -\frac{GM}{r^2}\dot{r}$$

$$\left(\frac{1}{2}(\dot{r})^2\right)^{\cdot} = \left(\frac{GM}{r}\right)^{\cdot} \quad \Big| \int_0^t$$

$$\frac{v^2}{2} - 0 = \frac{GM}{r} - \frac{GM}{h}$$

$$\frac{1}{2}v^2 = GM\left(\frac{1}{r} - \frac{1}{h}\right)$$

$$v(r) = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{h}\right)}$$

$$v_t = v(R) = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{h}\right)}$$

speciálně limita $h \to +\infty$ dává

$$v_t = \sqrt{\frac{2GM}{R}}$$

zjevně v rozporu s teorií relativity pro R malé

$$(v_t \ll c)$$

$$c \gg \sqrt{\frac{2GM}{R}} \to \qquad c^2 \gg \frac{2GM}{R}$$

$$\boxed{R \gg \frac{2GM}{c^2}}$$

potřebuji

$$\left(R_s := \frac{2GM}{c^2} \text{ je Schwarzschildův poloměr.} \right)$$

Homogenní rovnice

$\boxed{16.}$

$$y'(x+y) + x - y = 0$$

$$y' = \frac{y-x}{x+y} = \frac{\frac{y}{x} - 1}{1 + \frac{y}{x}}$$

$$u := \frac{y}{x}, \quad u' = \frac{y'}{x} - \frac{y}{x^2}$$

$\boxed{16.}$ $$y'(x+y)+x-y=0$$

$$y' = \frac{y-x}{x+y} = \frac{\frac{y}{x}-1}{1+\frac{y}{x}}$$

$$u := \frac{y}{x}, \quad u' = \frac{y'}{x} - \frac{y}{x^2}$$

$$y' - \frac{y}{x} = u'x$$

$$y' = u'x + u \longrightarrow xu' + u = \frac{u-1}{u+1}$$

$$xu' = \frac{u-1-u^2-u}{u+1} =$$

$$xu' = -\frac{u^2+1}{u+1}$$

$$\int \frac{du}{-\frac{u^2+1}{u+1}} = \int \frac{dx}{x}$$

$$-\int \frac{u+1}{u^2+1}\,du = \ln|x| + C$$

$$-\int \frac{u+1}{u^2+1}\,du = -\int \frac{\frac{1}{2}(u^2+1)'}{u^2+1}\,du - \int \frac{du}{u^2+1} =$$

$$= -\frac{1}{2}\ln(u^2+1) - \arctan(u^2+1) = \ln|x| + C$$

$$\rightarrow \boxed{\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\left(\frac{y^2}{x^2}+1\right) = \ln\left|\frac{x_0}{x}\right|}$$