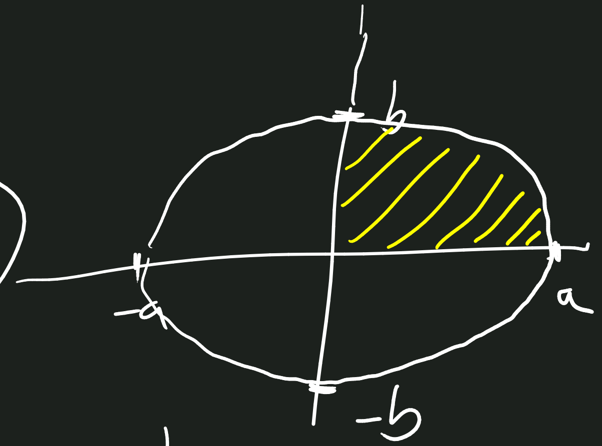
Aplikace určitého integrálu

Př. Určete obsah elipsy (poloosy $a \geq b$)

$$\varphi(t) = \begin{pmatrix} a \cos t \\ b \sin t \end{pmatrix}, \quad t \in [0, 2\pi]$$

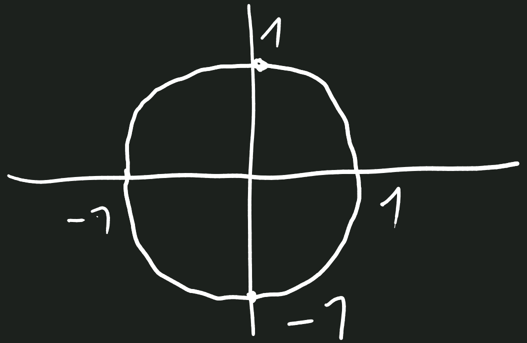
$$\boxed{S = \pi a b}$$



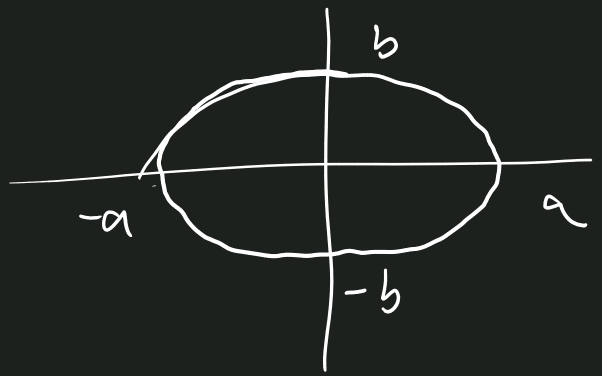
$$\frac{1}{4} S = \int_0^a y(x)\, dx \overset{\text{subs.}}{=} \left| \begin{array}{l} y(t) = b \sin t \\ x(t) = a \cos t \\ dx = -a \sin t \, dt \end{array} \right| =$$

$$= \int_0^{\pi/2} b \sin t \, (+a \sin t)\, dt = ab \int_0^{\pi/2} \sin^2 t \, dt = ab \int_0^{\pi/2} \frac{1 - \cos 2t}{2}\, dt =$$

$$= \underline{\frac{\pi}{4} ab}$$



$$L = \begin{pmatrix} a & \\ & b \end{pmatrix}$$



$S = \pi$

$S = (\det L)\,\pi$

$= \pi ab$

Obsah ohraničený křivkou v polárních souřadnicích

$$\varphi(\theta) = r(\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad \theta \in [0, 2\pi]$$

$$(r(\theta) > 0,\ \forall\theta)$$

Vzoreček pro obsah ohraničený touto křivkou

Uvažujme dělení D intervalu $[0, 2\pi]$

$\Delta\theta$

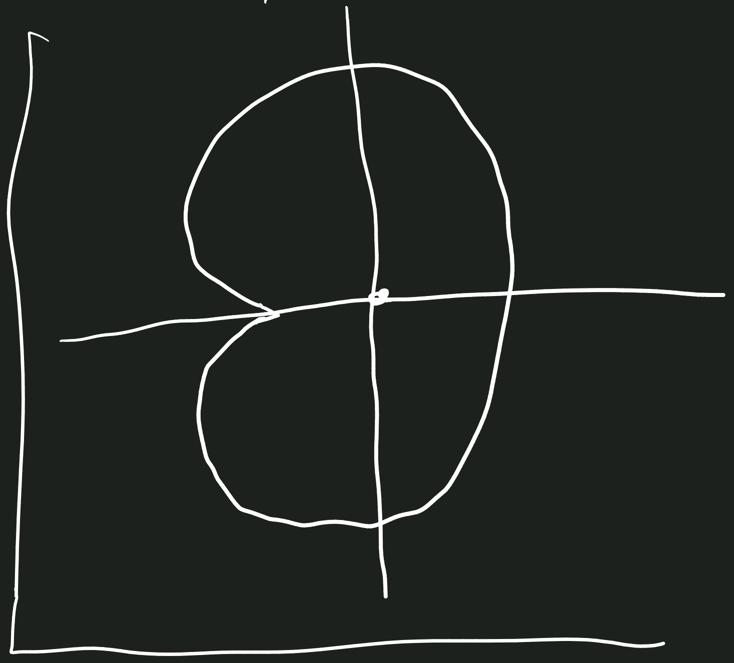
$$\sum \left(\frac{1}{2}\Delta\theta\, r_{min}^2 \le \Delta S \le \frac{1}{2}\Delta\theta\, r_{max}^2 \right.$$

$$\sup_D \frac{1}{2} s(r^2, D) \le S \le \inf_D \frac{1}{2} S(r^2, D)$$

$$\rightarrow \boxed{S = \int_0^{2\pi} \frac{r^2}{2}\, d\theta}$$

$\boxed{7}$ Obsah ohraničený kardiodou

$$r(\theta) = a(1 + \cos\theta), \quad a > 0, \quad \theta \in [0, 2\pi]$$



$$S = \frac{1}{2}\int_0^{2\pi} r^2(\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} a^2 (1 + \cos\theta)^2\, d\theta =$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta =$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta =$$

$$= \frac{a^2}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{a^2}{2} \cdot \frac{3}{2} \cdot 2\pi = \underline{\underline{\frac{3}{2}\pi a^2}}$$

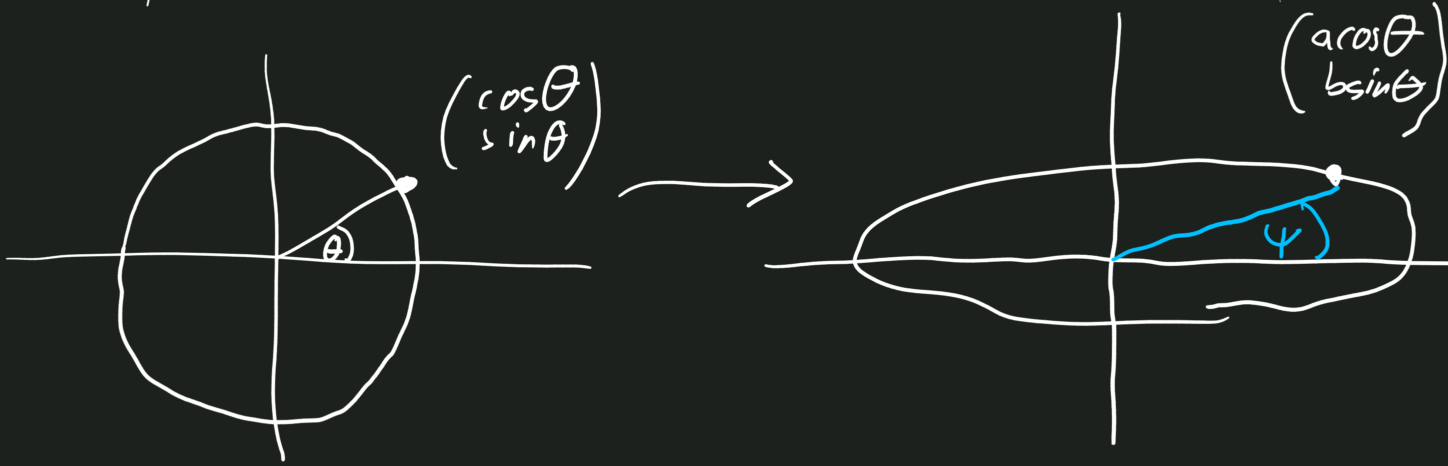
$$\varphi(\theta) = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi]$$

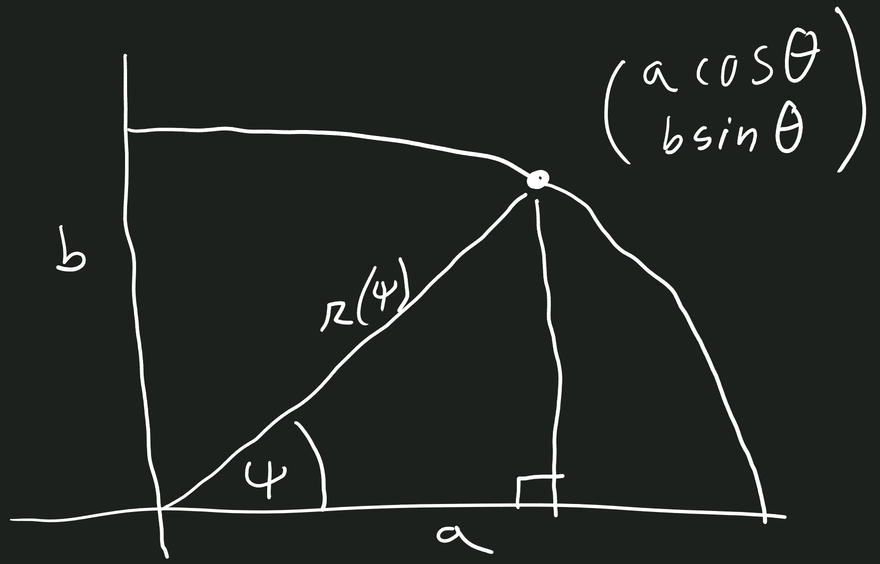
$$\boxed{\varphi(\theta) = r(\theta)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}$$

$$S = \frac{1}{2}\int_0^{2\pi} [r(\theta)]^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\theta =$$

$$= \frac{1}{2}\pi\left(a^2 + b^2\right) \neq \pi a b$$

$$\psi \neq \theta$$





$$\cos\psi = \frac{x}{r} = \frac{a}{r}\cos\theta \quad \Big| \cdot \frac{1}{a}$$

$$\sin\psi = \frac{y}{r} = \frac{b}{r}\sin\theta \quad \Big| \cdot \frac{1}{b}$$

$$\left(\frac{\cos\psi}{a}\right)^2 + \left(\frac{\sin\psi}{b}\right)^2 = \frac{1}{r^2}$$

$$\boxed{r = \frac{1}{\sqrt{\left(\frac{\cos\psi}{a}\right)^2 + \left(\frac{\sin\psi}{b}\right)^2}}}$$

$$\frac{1}{2}S = \int_0^{\pi/2} \frac{d\psi}{\left(\frac{\cos\psi}{a}\right)^2 + \left(\frac{\sin\psi}{b}\right)^2} =$$

$$= \left| \begin{array}{ll} t = \tan\psi \, , & dt = \dfrac{d\psi}{\cos^2\psi} \\ \cos^2\psi = \dfrac{1}{1+t^2} \, , & d\psi = \dfrac{dt}{1+t^2} \\ \sin^2\psi = \dfrac{t^2}{1+t^2} & \end{array} \right| = \frac{a^2}{a^2} \int_0^{+\infty} \frac{\frac{dt}{1+t^2}}{\frac{1}{a^2}\frac{1}{1+t^2} + \frac{1}{b^2}\frac{t^2}{1+t^2}} = a^2 \int_0^{+\infty} \frac{dt}{1 + \left(\frac{a}{b}t\right)^2}$$

$$= a^2 \left[\arctan\left(\frac{a}{b}t\right) \cdot \frac{b}{a}\right]_0^{+\infty} = \frac{\pi}{2}ab \longrightarrow \boxed{S = \pi ab}$$

$\boxed{8}$ určete obsah ohraničený lemniskátou

$$r(\theta) = 4\sin^2\theta, \quad 0 \le \theta \le 2\pi$$

$$S = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 16\, \overbrace{\sin^4\theta}^{\theta\text{ - periodická}}\, d\theta =$$

$$\overset{*}{=} 8\cdot 4\cdot \int_0^{\pi/2} \sin^4\theta\, d\theta = 32\, I_4 = 32\cdot\frac{3}{4} I_2 =$$

$$= 32\cdot\frac{3}{4}\cdot\frac{1}{2} I_0 = 32\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \underline{\underline{6\pi}}$$

$$\left(\int_0^{2\pi} \sin^4\theta\, d\theta = 2\int_0^{\pi} \sin^4\theta\, d\theta = 2\int_{-\pi/2}^{\pi/2} \sin^4\theta\, d\theta = 4\int_0^{\pi/2} \sin^4\theta\, d\theta \right)$$

Přestávka : pokračování ve 12:50

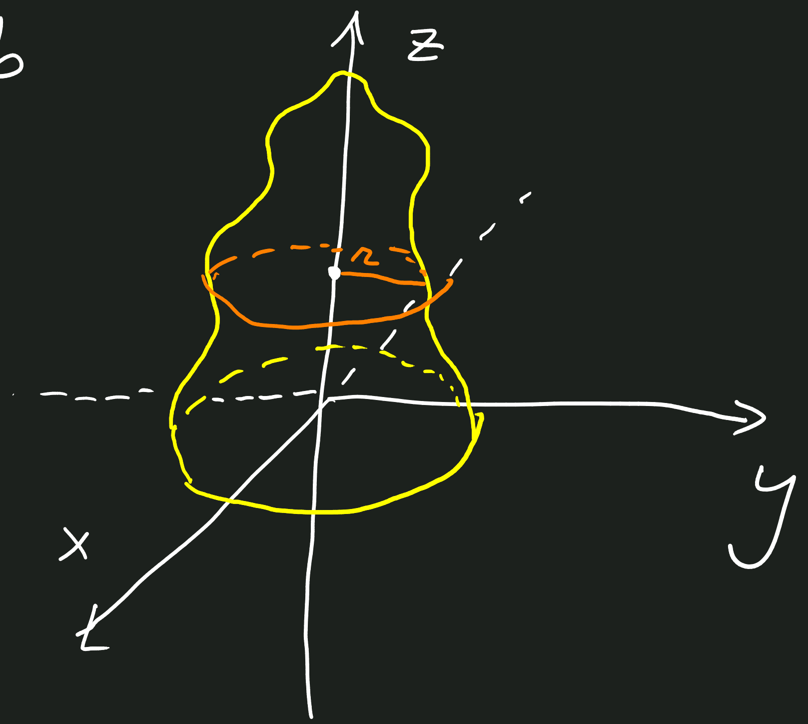
Objem & Povrch rotačního tělesa

$z \longmapsto r(z)$, $z \in [a,b]$, $a < b$

$M = \{ z \in [a,b],\ x^2 + y^2 \leq r(z)^2 \}$

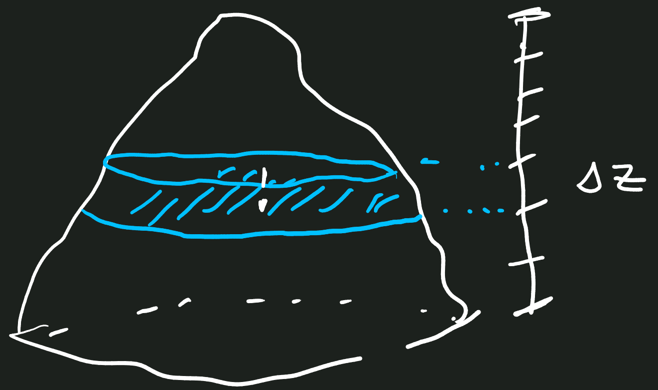
Q: Objem tělesa M ?

Q: Povrch tělesa M ?



1) Objem rotačního tělesa

Nechť D je dělení intervalu $[a,b]$



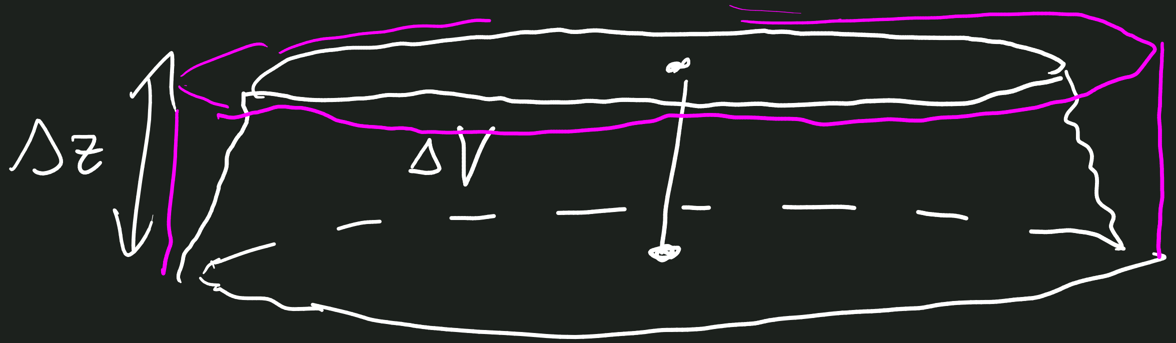
$$\pi r^2_{min} \cdot \Delta z \leq \Delta V \leq \pi r^2_{max} \cdot \Delta z$$

Σ

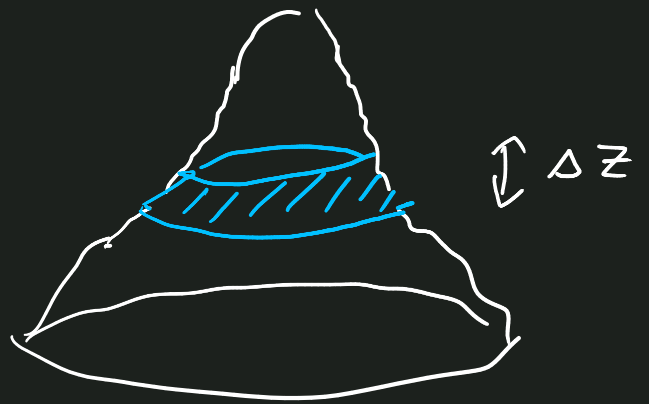
$$s(\pi r^2, D) \leq V \leq S(\pi r^2, D)$$

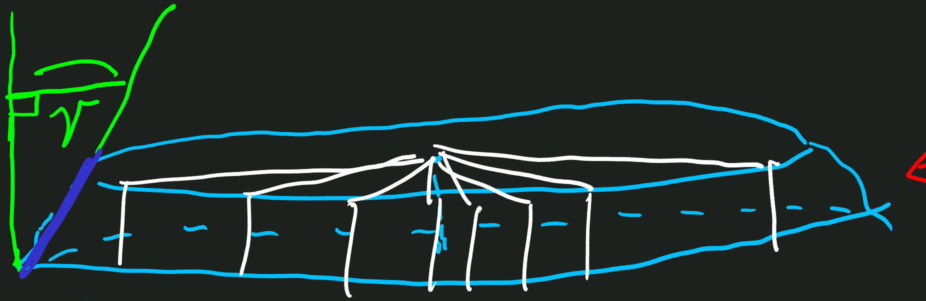
$$\sup_D s(\pi r^2, D) \leq V \leq \inf S(\pi r^2, D)$$

$$\Rightarrow \boxed{V = \int_a^b \pi r^2 \, dz}$$



2) Povrch rotačního tělesa





$$\Delta l = \frac{\Delta z}{\cos\gamma}$$

$$\Delta S = \frac{2\pi r \Delta z}{\cos\gamma} = 2\pi r \sqrt{1+\left(\frac{dr}{dz}\right)^2}\,\Delta z$$

$$\tan\gamma = -\frac{\Delta r}{\Delta z}$$

$$\gamma = -\arctan\left(\frac{dr}{dz}\right)$$

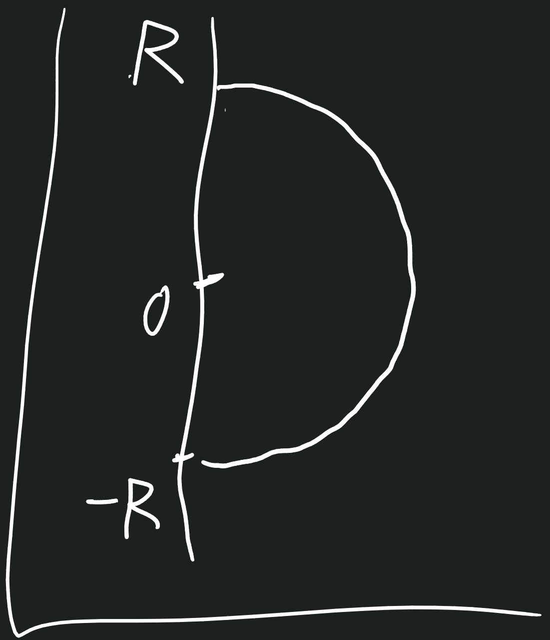
$$S = (r\Delta\theta)\cdot\Delta z$$

$$S = \frac{r\Delta\theta\Delta z}{\cos\gamma}$$

$$\Delta S = 2\pi r \sqrt{1 + \left(\frac{dr}{dz}\right)^2}\, \Delta z$$

$$\boxed{S = 2\pi \int_a^b r \sqrt{1 + \left(\frac{dr}{dz}\right)^2}\, dz}$$

Př. Objem a povrch koule



$$r(z) = \sqrt{R^2 - z^2}$$

$$V = \pi \int_{-R}^{R} r^2(z)\,dz = \pi \int_{-R}^{R} (R^2 - z^2)\,dz =$$

$$= \pi \left[R^2 z - \frac{z^3}{3} \right]_{-R}^{R} = 2\pi \left(R^3 - \frac{R^3}{3} \right) = \underline{\underline{\frac{4}{3} \pi R^3}}$$

$$S = 2\pi \int_{-R}^{R} \overbrace{r(z)}^{=\sqrt{R^2-z^2}} \sqrt{1+\left(\frac{dr}{dz}\right)^2}\, dz = 2\pi \int_{-R}^{R} \sqrt{R^2-z^2} \sqrt{1+\left(\frac{-\cancel{2}z}{\cancel{2}\sqrt{R^2-z^2}}\right)^2}\, dz$$

$$= 2\pi \int_{-R}^{R} \sqrt{R^2-z^2} \sqrt{1+\frac{z^2}{R^2-z^2}}\, dz = 2\pi \int_{-R}^{R} \sqrt{R^2-z^2+z^2}\, dz$$

$$= 2\pi \int_{-R}^{R} R\, dz = 2\pi R \cdot 2R = \underline{\underline{4\pi R^2}}$$

$$S' = 2\pi \int_{a}^{b} r(z) \sqrt{1+\left(\frac{dr}{dz}\right)^2}\, dz = 2\pi R \overbrace{(b-a)}^{h} = 2\pi R h$$

$$4\pi R^2 = \frac{d}{dR}\left(\frac{4}{3}\pi R^3\right)$$

tato formulka funguje JEN pro kouli

složitá tělesa při "nafukování" mění tvar

Délka křivky

$$\varphi(t): [a,b] \longrightarrow \mathbb{R}^n$$

$$L(\varphi) = \int_a^b \|\varphi'(t)\| \, dt$$

Délka kružnice

$$\varphi(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}, \quad t \in [0, 2\pi]$$

$$\varphi'(t) = R\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \quad \|\varphi'(t)\| = R$$

$$L(\varphi) = \int_0^{2\pi} R \, dt = \underline{\underline{2\pi R}}$$

$$\varphi(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}, \quad \varphi'(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}, \quad \|\varphi'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$L(\varphi) = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t} \, dt = \int_0^{2\pi} \sqrt{b^2 + (a^2 - b^2)\sin^2 t} = b\int_0^{2\pi} \sqrt{1 + \overbrace{\left(\frac{a^2}{b^2} - 1\right)}^{-\varepsilon^2} \sin^2 t}$$