Vázané extrémy

Věta: $U \subset \mathbb{R}^n$ otevřená

algebraická varieta →

$$M = \{ \vec{x} \in U \mid \vec{G}(\vec{x}) = 0 \}$$

kde $\vec{G} : U \to \mathbb{R}^m$

Nechť:

- $f : U \to \mathbb{R}$ má v bodě $\vec{x}_0 \in M$ lokální extrém vzhledem k M
- $\nabla f(\vec{x}_0)$ existuje
- $\vec{G}$ třídy C^1 na okolí bodu $\vec{x}_0$
- $\nabla G_1, \nabla G_2, \dots, \nabla G_m$ jsou lineárně nezávislé v $\vec{x}_0$

Potom:

$$\nabla f(\vec{x}_0) = \sum_{i=1}^{m} \lambda_i \nabla G_i(\vec{x}_0)$$

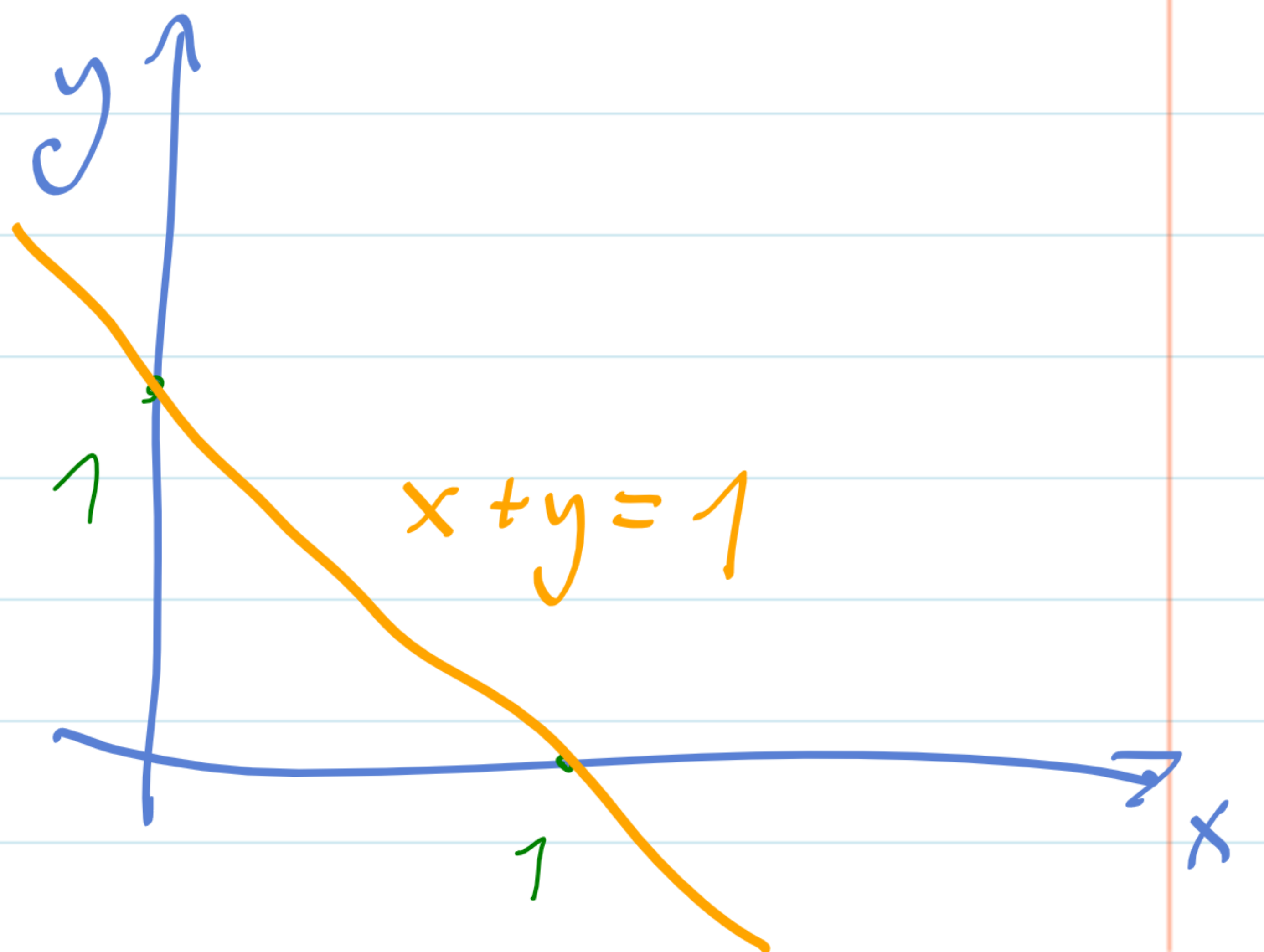
pro nějaká $\lambda_1, \lambda_2, \dots, \lambda_m$ (*Lagrangeovy multiplikátory*)

Př. $f(x,y) = xy$

extrémy f vzhledem k vazbě $x + y = 1$

$\Leftrightarrow$ $G(x,y) = x + y - 1$

$$M = \{ G(x,y) = 0 \}$$



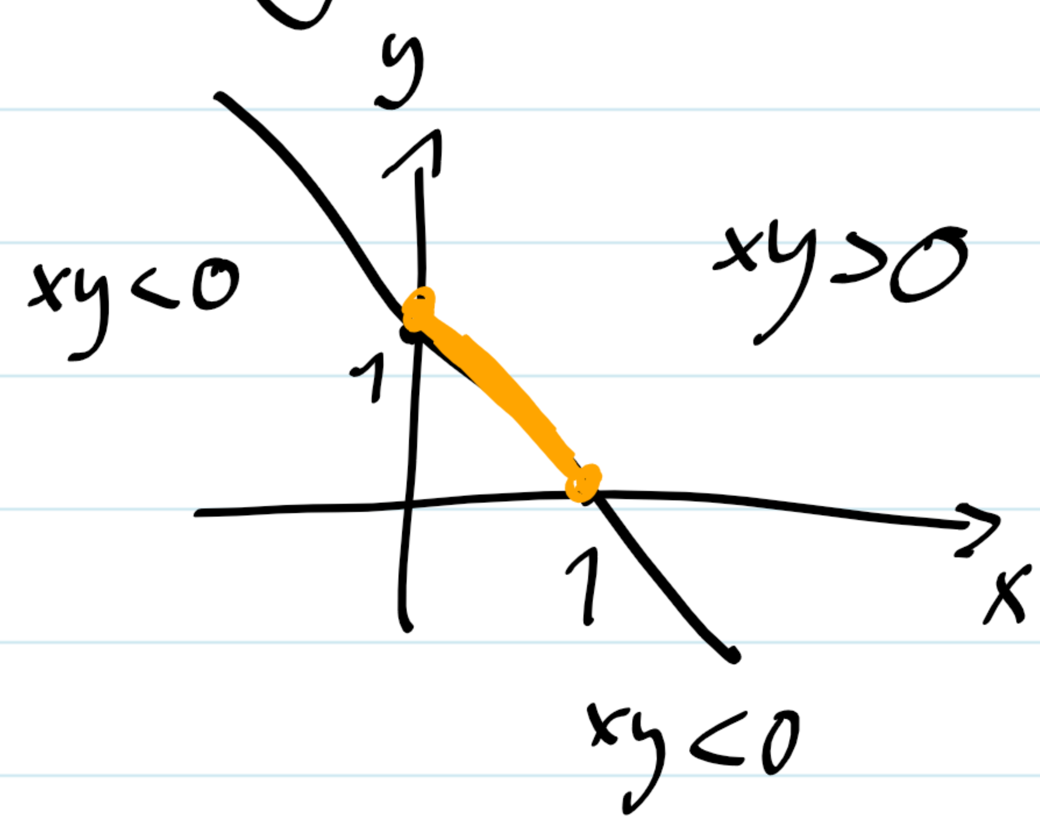
$$f(x,y) = xy$$
$$G(x,y) = x + y - 1$$

Pokud (x,y) je lokální extrém fce f na M, pak

$$\left.\begin{array}{l} \nabla f = \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ x + y = 1 \end{array}\right\} \quad x = y = \frac{1}{2}$$

Jediný bod $\left(\frac{1}{2}, \frac{1}{2}\right)$ je podezřelý z lok. extrému.

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$$



$$M = \{ (t, 1-t),\ t \in \mathbb{R} \}$$

$$f(t, 1-t) = t(1-t) = t - t^2 \longrightarrow -\infty, \text{ pro } t \to \infty$$

f na M nemá globální minimum

f zřejmě nabývá maxima někde v kvadrantu $xy \geq 0$

$\Longrightarrow$ $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$ je globální (i lokální) maximum fce f na M.

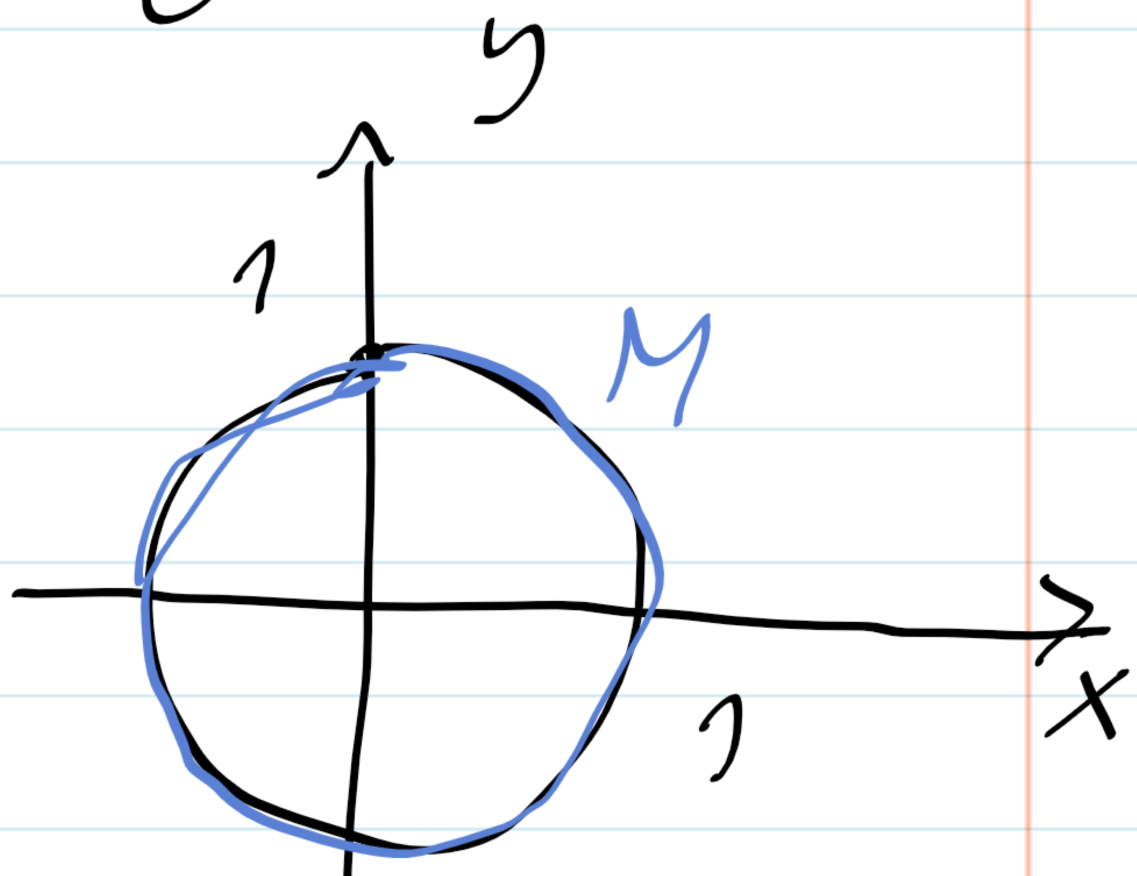
žádné jiné extrémy f na M nemá

2)

$$f(x,y) = \frac{x}{a} + \frac{y}{b}, \quad a,b > 0$$

$$G(x,y) = x^2 + y^2 - 1 = 0$$

$$M = \{G = 0\}$$



M je kompaktní

$\Rightarrow$ f nabývá na M glob. max. i min. (vázaného)

$$\nabla f = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix}, \quad \nabla G = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Je-li (x,y) lok. extrém, pak $\exists \lambda \in \mathbb{R}$:

$$\begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \quad x^2 + y^2 = 1$$

$$\frac{1}{a} = \lambda x, \quad \frac{1}{b} = \lambda y$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} = \lambda^2 \underbrace{(x^2 + y^2)}_{1} = \lambda^2$$

$$\lambda = \pm\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

$$x = \frac{1}{\lambda a}, \quad y = \frac{1}{\lambda b}$$

$\rightarrow$ Nechť $c > 0$, $\quad \frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Pak

$$\boxed{x = \pm\frac{c}{a}, \quad y = \pm\frac{c}{b}}$$

$$f\left(\frac{c}{a}, \frac{c}{b}\right) = \frac{c}{a^2} + \frac{c}{b^2} = \frac{1}{c} \longleftarrow \text{je max.}$$

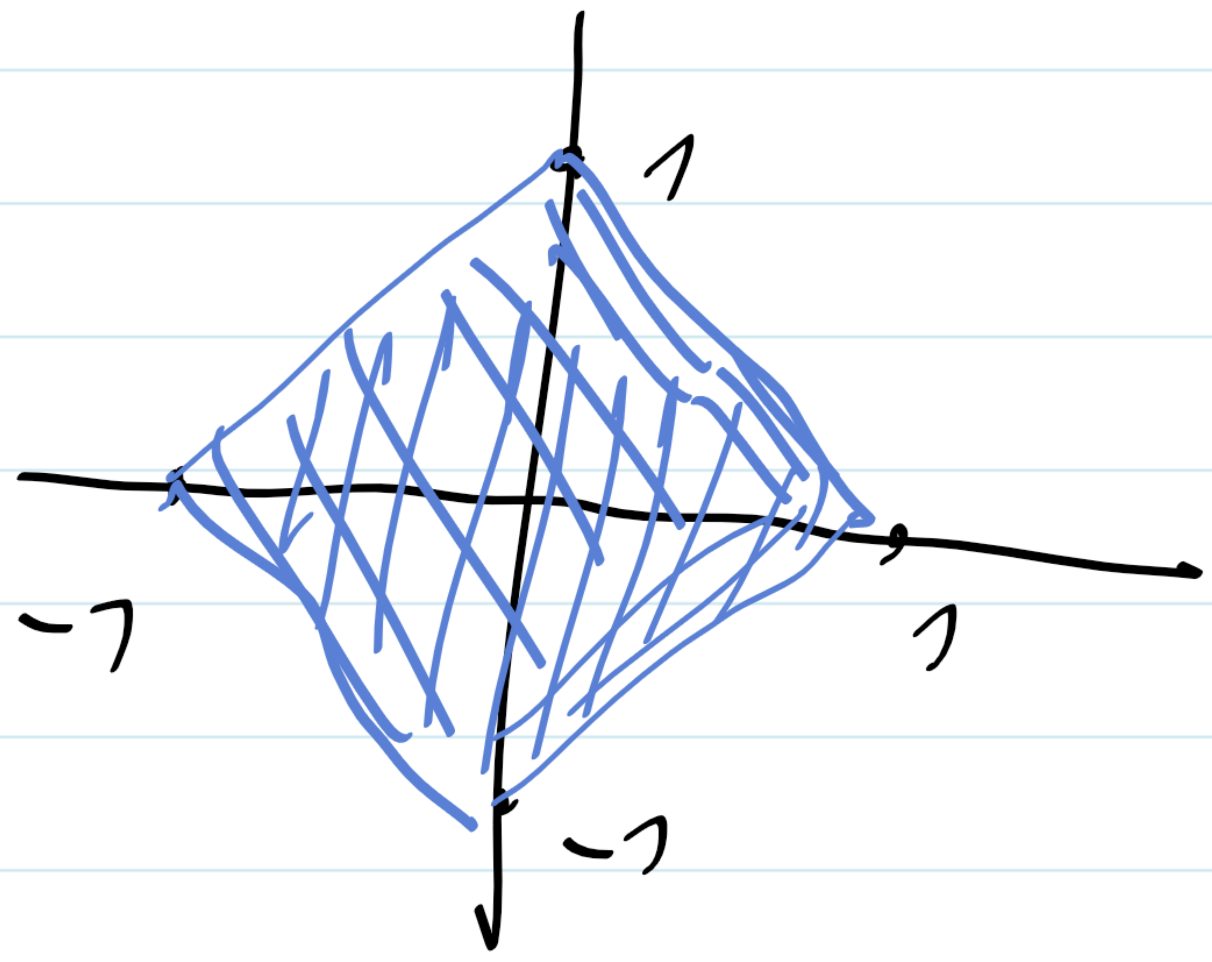
$$f\left(-\frac{c}{a}, -\frac{c}{b}\right) = -\frac{1}{c} \longleftarrow \text{je min.}$$

8) $f(x,y) = x^2 - xy + y^2$

nalzněte max i min fce f na množině

$$M = \{ |x| + |y| \leq 1 \}$$

M je kompaktní



- $M = \phi^{-1}((-\infty, 1])$

kde $\phi(x,y) = |x| + |y|$ je spojitá

$\rightarrow$ M je uzavřená

- $\forall (x,y) \in M: \quad |x| \leq 1, \quad |y| \leq 1$

$\rightarrow \sqrt{x^2+y^2} \leq \sqrt{2} \quad \Rightarrow$ M je omezená

$f(x,y) = x^2 - xy + y^2$ je spojitá

$\rightarrow$ f nabývá na M maxima i minima

(oko): $f(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1}{2}(x^2+y^2-2xy)$

$$= \frac{1}{2}(x^2+y^2) + \frac{1}{2}(x-y)^2 > 0 \text{ pro } (x,y) \neq (0,0)$$

$\wedge \quad f(0,0) = 0$

$\Rightarrow f(0,0) = 0$ je globální minimum

Nechť (x,y) je glob. maximum

i) Je-li $|x|+|y|<1$ $\quad ((x,y) \in \operatorname{Int} M)$

Potom

$$\nabla f(x,y) = 0$$

$$\rightarrow \begin{pmatrix} 2x-y \\ 2y-x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x=y=0$$

ale $f(0,0)=0$ je minimum ↯

ii) Je-li $|x|+|y|=1$, $\quad x \neq 0, y \neq 0$

Pak $\exists \lambda \in \mathbb{R}$:

$$\nabla f = \lambda \nabla \left(|x|+|y|-1 \right)$$

$$\begin{pmatrix} 2x-y \\ 2y-x \end{pmatrix} = \lambda \begin{pmatrix} \operatorname{sgn} x \\ \operatorname{sgn} y \end{pmatrix}$$

Nechť $x>0$, $y>0$

$$\left.\begin{aligned} 2x-y &= \lambda \\ 2y-x &= \lambda \end{aligned}\right\} +2 \qquad , \qquad x+y=1$$

$$3y = 3\lambda$$

$$y = \lambda = \frac{1}{2}$$

$$x = \lambda = \frac{1}{2}$$

Nechť $x>0$, $y<0$

$$\left.\begin{aligned} 2x-y &= \lambda \\ 2y-x &= -\lambda \end{aligned}\right\} +2 \qquad , \qquad \left|\frac{\lambda}{3}\right| + \left|\frac{\lambda}{3}\right| = 1$$

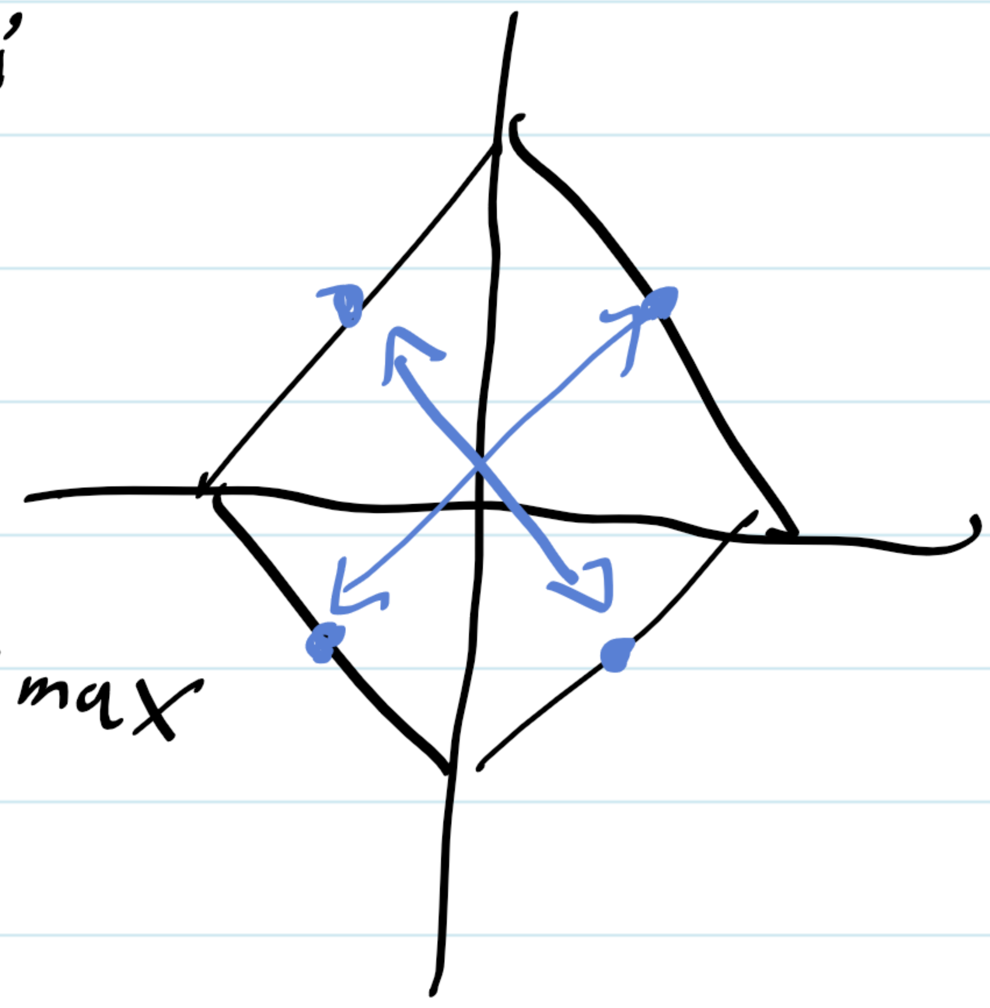
$$|\lambda| = \frac{3}{2}$$

$$3y = -\lambda \qquad \lambda = \frac{3}{2}$$

$$y = -\frac{\lambda}{3} \quad , \quad x = \frac{\lambda}{3} \qquad (x,y) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

$$f(x,y) = x^2 - xy + y^2 = f(-x,-y)$$

f má středovou symetrii



$$f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{4}$$

$$f\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{4}$$

$$f\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = f\left(-\tfrac{1}{2},\tfrac{1}{2}\right), \quad f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = f\left(-\tfrac{1}{2},-\tfrac{1}{2}\right)$$

iii)

$$f(0,1) = 1$$
$$f(1,0) = 1 \qquad 1 > \tfrac{3}{4}$$
$$f(0,-1) = 1$$
$$f(-1,0) = 1$$

$$\Rightarrow \quad \boxed{\max_{x \in M} f = 1, \quad \min_{x \in M} f = 0}$$

(15) Důkaz AG nerovnosti pomocí Lag. multiplikátorů

$$\underbrace{\frac{a_1 + \cdots + a_n}{n}}_{AM} \geq \underbrace{\sqrt[n]{a_1 \cdots a_n}}_{GM}$$

Pro $a_1, \ldots, a_n \geq 0$

Maximalizujme fci

$$f(x_1, \ldots, x_n) = x_1 \cdots x_n$$

s vazbou: $x_i \geq 0 \quad , \forall i$

$$G(\vec{x}) := x_1 + x_2 + \cdots + x_n = 1 \quad \Big\} M$$

M je kompaktní $\Rightarrow$ f má na M maximum

Nechť $\vec{x}$ je bod M, kde f má toto maximum

Pak $x_i > 0 \quad , \forall i$ (nebot' jinak $f = 0$)

$\Rightarrow \quad \exists \lambda \in \mathbb{R}: \quad \frac{\partial f}{\partial x_i} = \lambda \frac{\partial G}{\partial x_i} \quad , \forall i$

$\Rightarrow \quad \frac{f}{x_i} = \lambda$

$\Rightarrow \quad x_1 = x_2 = \cdots = x_n = \frac{1}{n}$

$\Rightarrow \quad \max_M f = f\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \frac{1}{n^n}$

$\forall \vec{x} \in M:$

$$f(\vec{x}) = \prod_{i=1}^{n} x_i \leq \frac{1}{n^n}$$

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n}$$

Definujme nyní $x_i = \frac{a_i}{n \cdot AM}$

Pak $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{a_i}{n} \cdot \frac{1}{AM}$

$= 1$ ✓

$$\sqrt[n]{\prod_{i=1}^{n} \frac{a_i}{n \cdot AM}} \leq \frac{1}{n}$$

$\rightarrow GM = \sqrt[n]{\prod_{i=1}^{n} a_i} \leq AM$

□