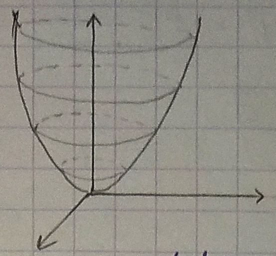
Funkce dvou (a více) proměnných

Př.$_1$ $f(x,y) = \dfrac{x^4+y^4}{x^2+y^2}$



→ $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$

– ale lze v $(0,0)$ spojitě dodefinovat? : $\lim\limits_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} 0$

* <u>def.</u>: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L \in \mathbb{R}$

$\Leftrightarrow$ — jakou normu def.? (typicky Eukl.) $\|(x,y)\| = \sqrt{x^2+y^2}$ Euklidovská norma

$\forall \varepsilon > 0 \; \exists \delta > 0 : \|(x,y)\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon$

→ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+y^4}{x^2+y^2} \overset{?}{=} 0$

nechť $\varepsilon > 0$: platí, že $\left|\dfrac{x^4+y^4}{x^2+y^2}\right| < \varepsilon$ pro malá $\sqrt{x^2+y^2}$?

Δ_1: $x^4+y^4 \leq x^4+2x^2y^2+y^4 = (x^2+y^2)^2$

$\mapsto$ $|f(x,y)| \leq \dfrac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2 = \|(x,y)\|^2$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Takže když $\delta = \sqrt{\varepsilon}$, $\|(x,y)\| < \delta \Rightarrow \|(x,y)\|^2 < \varepsilon$
$\Rightarrow |f(x,y)| < \varepsilon$

Takže $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Δ_2: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L \Leftrightarrow \lim\limits_{R\to 0^+} \left(\sup\limits_{\substack{\|(x,y)\| < R \\ (x,y) \neq 0}} |f(x,y) - L| \right) = 0$

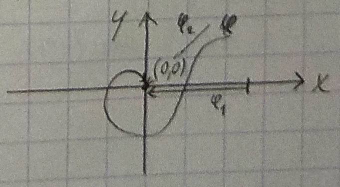
Př.$_2$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = ?$ — $f(x,y)$

→ z V o lim. slož. f-ce :

– pokud $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$,

$\varphi : (0,1) \to \mathbb{R}^2$ je křivka splňující $\lim\limits_{t\to 0^+} \varphi(t) = (0,0)$,

$\varphi(t) \neq (0,0) \quad \forall t \in (0,1)$



– potom $\lim\limits_{t\to 0^+} f \circ \varphi(t) = L$

i) volba$_1$ $\varphi_1(t) = (t,0)$

počítejme $\lim\limits_{t\to 0^+} f(t,0) = \lim\limits_{t\to 0^+} \underbrace{\dfrac{t\cdot 0}{t^2+0^2}}_{0} = 0$ $\Rightarrow$ pokud L $\exists$, potom $L = 0$

ii) volba$_2$ $\varphi_2(t) = (t,t)$

počítejme $\lim_{t\to 0^+} f(t,t) = \lim_{t\to 0^+} \frac{2t\cdot t}{t^2+t^2} = \lim_{t\to 0^+} 1 = 1$

$\Rightarrow$ pokud L $\exists$,
potom $L = 1$

$\Rightarrow$ $\lim_{(x,y)\to(0,0)} \overbrace{\frac{2xy}{x^2+y^2}}^{|.|\leq 1}$ neexistuje

když jdu lim. do nuly, záleží na směru

→ obecněji si mohu zvolit $\psi(t) = (at, t)$; $a \in \mathbb{R}$ parametr

$\lim_{t\to 0} f(at,t) = \frac{2at^2}{a^2t^2+t^2} = \frac{2a}{a^2+1}$ ⟿ lze dostat jako lim všelicos

$\boxed{5}$ $\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = ?$ ⟿ $f(x,y)$

→ $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$

→ předpokládejme, že L $\exists$ vlastní

pak$_1$ $L_1 = \lim_{t\to 0^+} f(t,0) = \lim_{t\to 0^+} \underbrace{(t^2)^0}_{1} = 1$ $\Rightarrow$ pokud L $\exists$ potom $L = 1$

pak$_2$ $L_2 = \lim_{t\to 0^+} f(t,t) = \lim_{t\to 0^+} (2t^2)^{t^4} = \lim_{t\to 0^+} \exp\left(\ln(2t^2)\cdot t^4\right) \overset{\text{volba}}{=} \exp(0) = 1$

→ $\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} \overset{?}{=} 1$

$\lim_{(x,y)\to(0,0)} \exp\left(\ln(x^2+y^2)\cdot x^2y^2\right) \overset{?}{=} 1$

z VOLFS$_{exp}$ stačí dokázat $\lim_{(x,y)\to(0,0)} x^2y^2 \ln(x^2+y^2) = 0$

stačí dokázat $\sup_{\substack{(x,y)\neq 0 \\ \|(x,y)\| < R}} \left|x^2y^2 \ln(x^2+y^2)\right| \xrightarrow{R\to 0^+} 0$

platí $\left|x^2y^2\ln(x^2+y^2)\right| \underset{\leq}{\overbrace{2|xy| \leq x^2+y^2}} \left(\frac{x^2+y^2}{2}\right)^2 \ln(x^2+y^2) \underset{\leq}{\overbrace{x^2+y^2 \leq R^2}}$

$\leq 2\frac{R^4}{4}\ln R \xrightarrow{R\to 0^+} 0$

takže skutečně $\underline{\underline{\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = 1}}$

11 $f(x,y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ $\rightarrow$

i) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = ?$ $= 0$

ii) $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x,y) = ?$ $= 0$

iii) $\lim\limits_{(x,y)\to 0} f(x,y) = ?$ $\nexists$

$D_f: \ x^2y^2 + (x-y)^2 = 0$

$D_f = \mathbb{R}^2 \setminus \{(0,0)\}$

i) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^2y^2}{x^2y^2+(x-y)^2}\right) = \lim\limits_{x\to 0} 0 = 0$

ii) jako i), ale $x \leftrightarrow y$ … $= 0$

iii) $\varphi(t) := \begin{pmatrix} at \\ bt \end{pmatrix}$; $a, b \in \mathbb{R}$

$\lim\limits_{t\to 0^+} f(at, bt) = \lim\limits_{t\to 0^+} \dfrac{a^2b^2t^4}{a^2b^2t^4 + (a-b)^2t^2}$

$\overset{a=b}{=} \lim\limits_{t\to 0^+} \dfrac{a^2}{a^2+0} = 1$

$\overset{a=0,\ b\neq 0}{=} \quad = 0$

$\Rightarrow \lim \nexists$

- **Parciální derivace & totální diferenciál**

$f(x,y)$ ……… derivace ? $\quad * \ f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(I) **Parciální derivace** $\partial_x f$, $\partial_y f$

jako v $\mathbb{R}^1$, druhou proměnnou zafixuji

(II) **Totální diferenciál** $df_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}$

lineární zobrazení takové, že platí:

$$f(x+dx, y+dy) = f(x,y) + df_{(x,y)}(dx, dy) + o(\overbrace{\|(dx,dy)\|}^{\sqrt{dx^2+dy^2}})$$

(i) existuje-li $df_{(x,y)}$, pak existují $\partial_x f, \partial_y f$ v bodě (x,y), $df_{(x,y)}(dx,dy) = \partial_x f \cdot dx + \partial_y f \cdot dy = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}$

(ii) je-li U otevřená množina a $\partial_x f, \partial_y f \in C(U)$, potom existuje df v U

$\boxed{8}$ $f(x,y) = x^4 + y^4 - 4x^2y^2$ $\rightarrow$ vyšetřete $\partial_x f$, $\partial_y f$, df

$$\left.\begin{aligned} \rightarrow \quad \partial_x f(x,y) &= 4x^3 + 0 - 8xy^2 \\ &= 4x\,(x^2 - 2y^2) \\ \rightarrow \quad \partial_y f(x,y) &= 0 + 4y^3 - 8x^2y \\ &= 4y\,(y^2 - 2x^2) \end{aligned}\right\} \quad \forall (x,y) \in \mathbb{R}^2$$

$\triangle$: $\partial_x f$, $\partial_y f$ jsou $C(\mathbb{R}^2)$

$\Rightarrow$ $df \; \exists \quad \forall (x,y) \in \mathbb{R}^2 : \; df_{(x,y)}(dx,dy) = 4x(x^2-2y^2)\,dx + 4y(y^2-2x^2)\cdot dy$; $dx, dy \in \mathbb{R}$

* význam: $\forall (x,y) \in \mathbb{R}^2$ je $\begin{pmatrix} x' = x + dx \\ y' = y + dy \end{pmatrix}$

$\ell(x',y') = f(x,y) + df_{(x,y)}(x'-x,\, y'-y)$ je

nejlepší lineární aproximace f-ce $f(x',y')$ na okolí

bodu (x,y)

(je to tečná rovina)

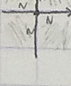
$\boxed{3}$ $f(x,y) = |xy|$

$\rightarrow$ f spojitá na $\mathbb{R}^2$

$\rightarrow$ · pro $(x,y) \in M = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge y \neq 0\}$

$$\left.\begin{aligned} \partial_x f &= (\operatorname{sgn} x)\,|y| \\ \partial_y f &= (\operatorname{sgn} y)\,|x| \end{aligned}\right\} \text{ spojité na } \overset{\text{otevřená}}{M} \;\Rightarrow\; \begin{aligned} &df_{(x,y)}(dx,dy) = (\operatorname{sgn} x)|y|\,dx + (\operatorname{sgn} y)|x|\,dy \\ &\forall (x,y) \in M \end{aligned}$$

sgn obecně nespojitá, zde však je spojitá (body nespojit. jsou vyloučeny) → tj. osy x, y



· $N = \{(x,y) \in \mathbb{R}^2 : x = 0 \;\underline{\vee}\; y = 0\}$

$\forall (x,y) \in N$: $\partial_x f$ nebo $\partial_y f$ $\nexists$ $\left(x = 0 \;\Rightarrow\; \begin{aligned} &\partial_x f \nexists \\ &\partial_y f = 0 \end{aligned} \quad \text{a } \circlearrowleft \right)$

$\Rightarrow$ na mn. N $\nexists\, df$ (totální dif.)

· $(x,y) = 0 \;\rightarrow\; f(dx,dy) = |dx\,dy| \leq \frac{1}{2}(dx^2 + dy^2) = o\left(\sqrt{dx^2 + dy^2}\right)$

$\Rightarrow$ $df_{(0,0)} = 0$

$\Rightarrow$ $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$

$\boxed{16}_{10}$ $f(x,y) = x^2 - xy + y^2$; $(x,y) \in \mathbb{R}^2$
$(x_0, y_0) = (1,1)$

→ najděte jednotkový vektor $\vec{v} \in \mathbb{R}^2$ takový, že

$\frac{\partial f}{\partial \vec{v}}(1,1)$ je a) největší
b) nejmenší
c) nulová

$f(x,y) = x^2 - xy + y^2$

$\frac{\partial f}{\partial x} = 2x - y$; $\frac{\partial f}{\partial y} = -x + 2y$

→ f má totální diferenciál

reprezentován gradientem $\nabla f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix}$

$\nabla f(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \frac{\partial f}{\partial \vec{v}}(1,1) = \vec{v} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

⇒ a) maximální pro $\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) minimální pro $\vec{v} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) nulový pro $\vec{v} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

* pro $\left| \vec{v} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right| \leq \|\vec{v}\| \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2}$

pro $\vec{v} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ nastane rovnost

$\boxed{17}_{10}$ $f(x,y) = xy \frac{x^2 - y^2}{x^2 + y^2}$ → i) limita v 0
ii) parciální derivace $\partial_x f$, $\partial_y f$, $\partial^2_{xy} f$, $\partial^2_{yx} f$

Ad i) $\lim_{t \to 0} f(t,0) = 0$

$\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta) = r^2 \cos\theta \sin\theta \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$\lim_{r \to 0^+} \left(\sup_{\theta \in [0, 2\pi)} \left| \tilde{f}(r,\theta) \right| \right) = 0$?

$\left| \tilde{f}(r,\theta) \right| = r^2 \overbrace{|\cos\theta \sin\theta|}^{\leq 1} \cdot \overbrace{|\cos^2\theta - \sin^2\theta|}^{\leq 2}$

$\leq 2r^2 \xrightarrow{r \to 0^+} 0$

Ad ii) $\frac{\partial f}{\partial x} = y \frac{(3x^2 - y^2)(x^2 + y^2) - (x^3 - xy^2)(2x)}{(x^2 + y^2)^2} =$

$= y \frac{3x^4 + 2x^2y^2 - y^4 - 2x^4 + 2x^2y^2}{(x^2 + y^2)^2} =$

$\frac{\partial f}{\partial x} = y \frac{x^4 + 4x^2y^2 - y^4}{(x^2 + y^2)^2}$; $(x,y) \neq (0,0)$

navíc zřejmě: $\frac{\partial f}{\partial x}(0,0) = 0$

$$f(x,y) = x\,\frac{x^2y-y^3}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = x\,\frac{(x^2-3y^2)(x^2y^2)-(x^2y-y^3)(2y)}{(x^2+y^2)^2} = x\,\frac{x^4-2x^2y^2-3y^4-2x^2y^2+2y^4}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = x\,\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2} \quad ; \quad (x,y) \neq (0,0)$$

navíc zřejmě: $\frac{\partial f}{\partial y}(0,0) = 0$

$$\frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)\bigg|_{(0,0)} = \lim_{x\to 0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x\to 0}\frac{\frac{x^5}{x^4}-0}{x} = 1$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)\bigg|_{(0,0)} = \lim_{y\to 0}\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y\to 0}\frac{\frac{-y^5}{y^4}-0}{y} = -1$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x\,\partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\,\partial x}(0,0)$$

$\boxed{\text{Př.}}$ spočtěte tot. dif. f-ce g ; $g(u,v) = f(u^2+v^2, u^2-v^2, 2uv)$
$f = f(x,y,z)$ je f-ce třídy C^1

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

(řetízkové pravidlo)

$$= 2u\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y} + 2v\frac{\partial f}{\partial z}$$

$*\quad dg(du,dv) = \nabla g \cdot \begin{pmatrix} du \\ dv \end{pmatrix}$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$= 2v\frac{\partial f}{\partial x} - 2v\frac{\partial f}{\partial y} + 2u\frac{\partial f}{\partial z}$$

$$dg(du,dv) = \left(2u\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right] + 2v\frac{\partial f}{\partial z}\right)du + \left(2v\left[\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right] + 2u\frac{\partial f}{\partial z}\right)dv$$

Implicitní funkce

→ **VOIF**: Zobr. $f = f(\vec{x}, y) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$
třídy $C^k(U)$; U je okolí bodu $(\vec{x}_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$
a nechť $\partial_y f(\vec{x}_0, y_0) \neq 0$

Potom $\exists V$ okolí bodu x_0 $\forall x \in V$ $\exists! y$: $(\vec{x}, y) \in U$, $f(\vec{x}, y) = 0$
Navíc f-ce $y = y(\vec{x})$ je třídy $C^k(V)$

$\boxed{11}_{12}$ D, že $\exists V$ okolí b. $(1,1)$: $\{(x,y) ; x^3 + y^3 - 2xy = 0\} \cap V$ je grafem f-ce $y = y(x)$ třídy C^2 na nějakém okolí 1

spočtěte $y'(1)$, $y''(1)$

$f(x,y) = x^3 + y^3 - 2xy$; $(x,y) \in \mathbb{R}^2$, $f \in C^\infty(\mathbb{R}^2)$

$\frac{\partial f}{\partial y}(x,y) = 3y^2 - 2x \overset{(x,y)=(1,1)}{=} 1 \neq 0$

$\overset{\text{VOIF}}{\Longrightarrow}$ $\exists V$ okolí bodu 1 takové, že $\forall x \in V$ $\exists! y : f(x,y) = 0$
& f-ce $y = y(x)$ je třídy C^∞

$f(x, y(x)) = 0$, $\forall x \in V$

$$\to \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0 \quad \Rightarrow \quad y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 2x \xrightarrow{(x,y)=(1,1)} \underline{\frac{\partial f}{\partial y}(1,1) = 1}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^3 - 2xy\right) = 3x^2 - 2y \xrightarrow{(x,y)=(1,1)} \underline{\frac{\partial f}{\partial x}(1,1) = 1}$$

$\Rightarrow$ $\underline{y'(1) = -1}$

$$\frac{d}{dx}\left(\overbrace{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'}^{0}\right) = 0$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x} y' + \frac{\partial^2 f}{\partial x \partial y} y' + \frac{\partial^2 f}{\partial y^2}(y')^2 + \frac{\partial f}{\partial y} y'' = 0$$

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} y' + \frac{\partial^2 f}{\partial y^2}(y')^2 + \boxed{\frac{\partial f}{\partial y}} y'' = 0$$

$$\to \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3y^2 - 2x\right) = -2$$

$$\to \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 2y\right) = 6x$$

$$\to \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(3y^2 - 2x\right) = 6y$$

$$\Rightarrow \quad 6 + 2 \cdot (-2) \cdot (-1) + 6 \cdot (-1)^2 + 1 \cdot y'' = 0$$
$$6 + 4 + 6 + y'' = 0$$

$\Rightarrow$ $\underline{y''(1) = -14}$