ORGANIZACE

3 zápočtové testy $\quad 3 \times 10$ bodů

1. test : 16. 4. (dif. rovnice + urč. integrálů)
2. test : 7. 5. (řady)
3. test : 4. 6. (funkce více proměnných)

Bonusové body : max 10 bodů

5 sérií, 2 příklady, každý 1 bod

$5 \times 2 \times 1 = 10$ bodů

$\boxed{\text{K získání zápočtu 15 bodů}}$

Newtonův integrál

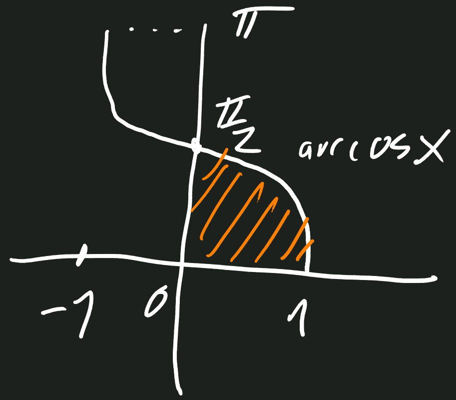
Př.
$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \left| \begin{array}{l} \text{sub.} \\ e^x - 1 = t^2 \\ t = \sqrt{e^x - 1} \\ x = \ln(t^2+1) \end{array} \quad \begin{array}{l} x_0 = 0, \; x_1 = \ln 2 \\ t_0 = 0, \; t_1 = 1 \\ dx = \frac{2t}{t^2+1}\, dt \end{array} \right.$$

$$= \int_0^1 t \, \frac{2t\, dt}{t^2+1} = \int_0^1 \frac{2t^2 + 2 - 2}{t^2+1}\, dt =$$

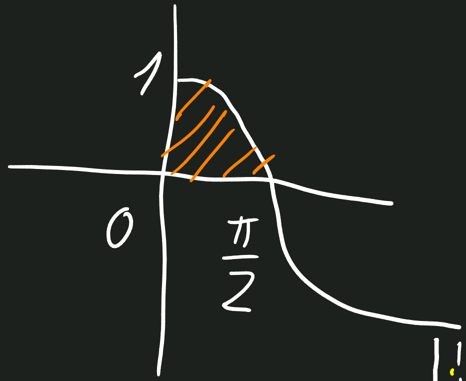
$$= \int_0^1 \left(2 - \frac{2}{t^2+1}\right) dt = \Big[2t - 2\operatorname{arctan} t\Big]_0^1 =$$

$$= (2 - 2\operatorname{arctan} 1) - 0 = \boxed{2 - \frac{\pi}{2}}$$

2.) $$\int_0^1 \underset{\downarrow\, -\frac{1}{\sqrt{1-x^2}}}{\overset{x\,\uparrow}{1}\cdot\arccos x}\,dx = 1$$

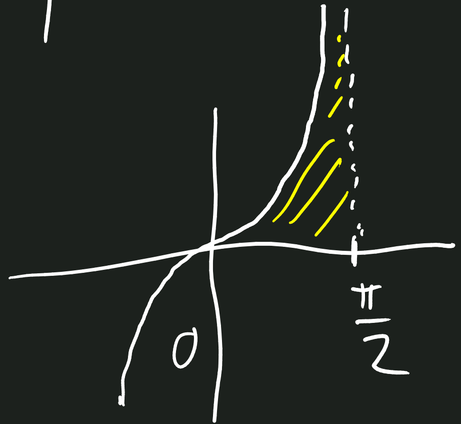


10.) $$\int_0^{\pi/2} \tan x \, dx = +\infty$$



$$\int_{-\pi/2}^{\pi/2} \tan x \, dx = N/A$$

$$\cancel{\lim_{\alpha \to \frac{\pi}{2}^-} \int_{-\alpha}^{\alpha} \tan x\,dx}$$



$$\int_0^1 1\cdot \arccos x \, dx = \left| \begin{array}{ll} u = \arccos x \;, & u' = -\frac{1}{\sqrt{1-x^2}} \\ v' = 1 \;, & v = x \end{array} \right| =$$

$$= \Big[x \arccos x \Big]_0^1 - \int_0^1 \left(- \frac{x}{\sqrt{1-x^2}} \right) dx =$$

$$= 0 + \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx = \Big[- \sqrt{1-x^2} \Big]_0^1 = 1$$

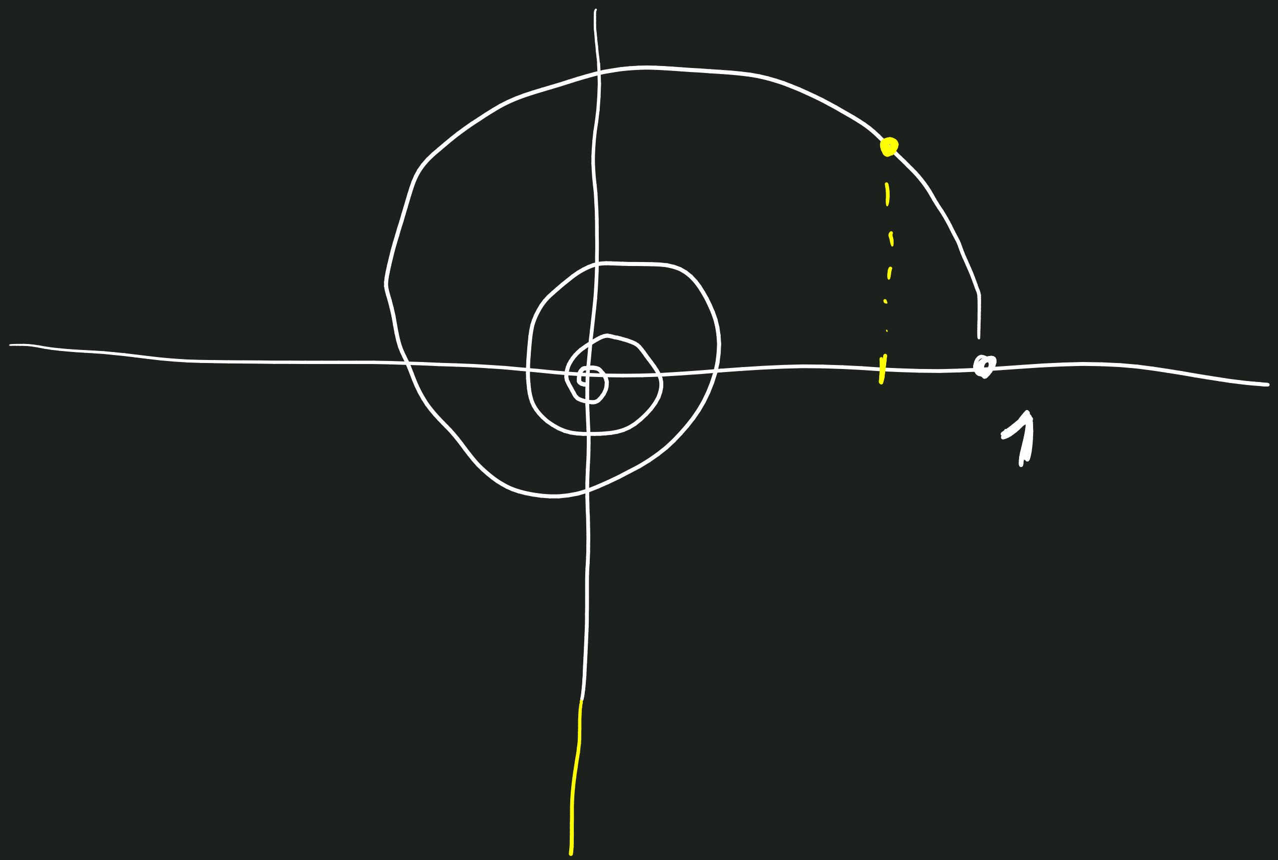
g) $$\int_0^{+\infty} e^{-ax}\cos bx \, dx \quad (a>0) = \mathrm{Re}\left(\int_0^{+\infty} e^{-ax}(\cos bx + i\sin bx)\, dx\right)$$

$$= \mathrm{Re}\left(\int_0^{+\infty} e^{-ax+ibx}\, dx\right) =$$

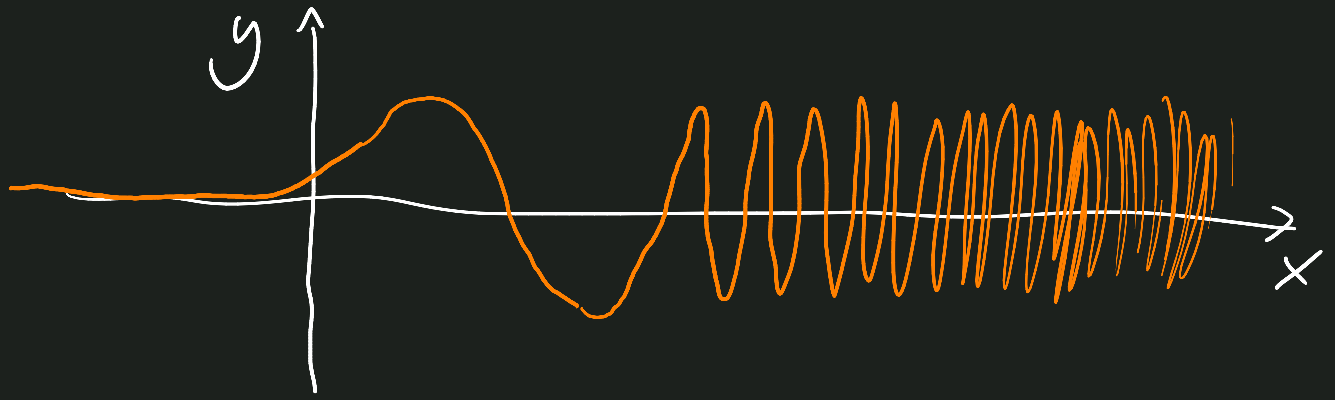
$$= \mathrm{Re}\left(\int_0^{+\infty} \frac{d}{dx}\left[\frac{e^{(-a+ib)x}}{-a+ib}\right] dx\right)$$

$$= \mathrm{Re}\left(\left[\frac{e^{(-a+ib)x}}{-a+ib}\right]_0^{+\infty}\right) = 0 - \mathrm{Re}\left(\frac{1}{-a+ib}\right)$$

$$= \mathrm{Re}\left(\frac{1}{a-ib}\right) = \mathrm{Re}\left(\frac{a+ib}{a^2+b^2}\right) = \frac{a}{a^2+b^2}$$


1

$$\int_{-\infty}^{+\infty} \sin(e^x)\,dx = \left| \begin{array}{l} e^x = t \\ e^x\,dx = dt \\ dx = \frac{dt}{t} \end{array} \right| = \int_0^{+\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$



souvislost ?

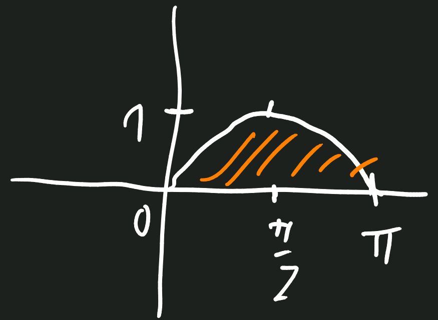
$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

$$e^x,\ \sin x,\ x^2 + x,\ \frac{1}{x+1},\ \dots$$

Wallisovy integrály

$$I_n = \int_0^{\pi} \sin^n x \, dx, \quad n = 0, 1, 2, 3, \dots$$

$$I_0 = \int_0^{\pi} 1 \, dx = \pi$$



$$I_1 = \int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi} = -(-1) - (-1) = 2$$

$\forall n \in \mathbb{N}_0$: $I_n \geq 0$, $\quad I_n \geq I_{n+1}, \ \forall n$

$$I_{n+2} = \int_0^{\pi} \sin^{n+1} x \cdot \sin x \, dx$$

$$I_{n+2} = \int_0^{\pi} \sin^{n+1} x \cdot \sin x \, dx = \underbrace{\left[\sin^{n+1} x \, (-\cos x) \right]_0^{\pi}}_{0} -$$

$$- \int_0^{\pi} (n+1) \sin^n x \, (+\cos x)(-\cos x) \, dx =$$

$$= +(n+1) \int_0^{\pi} \overset{1-\sin^2 x}{\cos^2 x} \sin^n x \, dx = +(n+1) \int_0^{\pi} \sin^n x \, dx -$$

$$-(n+1) \int_0^{\pi} \sin^{n+2} x \, dx = +(n+1) I_n - (n+1) I_{n+2}$$

$$I_{n+2} = +(n+1) I_n - (n+1) I_{n+2}$$

$$I_{n+2} = +(n+1) I_n - (n+1) I_{n+2}$$

$$(n+2) I_{n+2} = (n+1) I_n$$

$$I_{n+2} = \boxed{\left(\frac{n+1}{n+2}\right)} I_n \quad , \quad \forall n \in \mathbb{N}_0$$

$$I_0 = \pi \quad , \quad I_2 = \pi \cdot \frac{1}{2} \quad , \quad I_4 = \pi \cdot \frac{1}{2} \cdot \frac{3}{4}$$

$$I_1 = 2 \quad , \quad I_3 = 2 \cdot \frac{2}{3} \quad , \quad I_5 = 2 \cdot \frac{2}{3} \cdot \frac{4}{5}$$

$$I_{n+2} \leq I_{n+1} \leq I_n$$

$$I_{n+2} \le I_{n+1} \le I_n \quad \Big| : I_n$$

$$\frac{n+1}{n+2} = \frac{I_{n+2}}{I_n} \le \frac{I_{n+1}}{I_n} \le 1 \quad \Big| \lim_{n\to\infty}$$

$$\frac{n+1}{n+2} \to 1, \qquad 1 \to 1$$

$$\lim_{\substack{n\to\infty \\ n \text{ sudá}}} \frac{I_{n+1}}{I_n} = 1$$

$$\lim_{n\to\infty} \frac{2 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n}{2n+1}}{\pi \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n}} = 1$$

$$\lim_{n\to\infty} \frac{2\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n+1}}{\pi\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{2n-1}{2n}} = 1 \quad \Big|\cdot\frac{\pi}{2}$$

$$\lim_{n\to\infty} \frac{(2\cdot 4\cdot\ldots\cdot 2n)^2}{(1\cdot 3\cdot 5\cdots 2n-1)^2\,(2n+1)} = \frac{\pi}{2}$$

$$\lim_{n\to\infty} \frac{2^{2n}\,(\overbrace{1\cdot 2\cdot 3\cdots n}^{n!})^2}{(1\cdot 3\cdot 5\cdots(2n-1))^2\,(2n+1)}\cdot\frac{(2\cdot 4\cdots 2n)^2}{(2\cdot 4\cdots 2n)^2}$$

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{2n}\,(\overbrace{1\cdot 2\cdot 3\cdots n}^{n!})^2}{(\underbrace{1\cdot 3\cdot 5\cdots(2n-1)}_{})^2\,(2n+1)} \cdot \frac{(2\cdot 4\cdots 2n)^2}{(\underbrace{2\cdot 4\cdots 2n}_{})^2}$$

$$= \lim_{n\to\infty} \frac{\left(2^{2n}\,(n!)^2\right)^2}{(2n!)^2\,(2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{(2n+1)\binom{2n}{n}^2}$$

$$\sqrt{\frac{2}{\pi}} = \lim_{n\to\infty} \frac{\sqrt{2n+1}\,\binom{2n}{n}}{4^n}$$

Wallisův součin

$$\boxed{\frac{1}{2^{2n}}\binom{2n}{n} \sim \sqrt{\frac{2}{\pi(2n+1)}}}$$

Stirlingův vzorec

$$\boxed{n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}$$

Konvergence určitých integrálů

f def na (a,b), spojitá

$$\int_a^b f\,dx = F(b^-) - F(a^+)$$

hledání PF $>$ výpočet $\int_a^b$ $>$ konvergence $\int_a^b$

Otázka:

i) $\int_a^b f\,dx$ existuje v $\mathbb{R}$ ... konverguje

ii) $F(b^-) = +\infty,\ F(a^+) = 0$ .... diverguje do $+\infty$ (resp $-\infty$)

iii) PS nemá smysl ... integrál nemá smysl

absolutní konvergence

$\int_a^b f(x)\,dx$ konverguje absolutně, jestliže $\int_a^b |f(x)|\,dx$ konverguje

Tvrzení: když $\int_a^b f(x)\,dx$ konverguje absolutně (AK)

$\Rightarrow$ $\int_a^b f(x)$ konverguje

Srovnávací kritérium: nechť f, g jsou spojité na $[a, b)$

$\int_a^b |f|\,dx \le g$

a nechť $|f| \le g$ na $[a, b)$

a nechť $\int_a^b g\,dx$ konverguje

Pak $\int_a^b f\,dx$ konverguje absolutně.

Limitní srovnávací kritérium

Jsou-li f, g spojité a nezáporné na $[a,b)$, kde $-\infty < a < b \leq +\infty$

a platí-li

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = c \in \mathbb{R} \setminus \{0\}$$

Pak platí:

$\int_a^b f\,dx$ konverguje $\iff$ $\int_a^b g\,dx$ konverguje.

15)
$$\int_0^{+\infty} \underbrace{\frac{x^{3/2}}{1+x^2}}\,dx = \int_0^1 \frac{x^{3/2}}{1+x^2}\,dx + \int_1^{+\infty} \frac{x^{3/2}}{1+x^2}\,dx$$

$f(x) \geq 0, \; \forall x \in (0, +\infty)$

$f(x) = |f(x)|$

$\underline{\text{v } 0^+}$
$$\frac{x^{3/2}}{1+x^2} \sim x^{3/2}, \; x \to 0^+$$

$\underline{\text{v } +\infty}$
$$\frac{x^{3/2}}{1+x^2} \sim x^{3/2-2} = x^{-1/2}, \; x \to +\infty$$

Podle limitního srovnávacího kritéria:

$$\int_0^1 \frac{x^{3/2}}{1+x^2}\,dx \; K \iff \int_0^1 x^{3/2}\,dx \; K$$

$$\int_0^1 x^{3/2}\,dx = \left[\frac{x^{5/2}}{\frac{5}{2}}\right]_0^1 = \frac{2}{5}$$

$$\int_1^{+\infty} \frac{x^{3/2}}{1+x^2}\,dx \; K \iff \int_1^{+\infty} x^{-1/2}\,dx \; K$$

$$= \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_1^{+\infty} = +\infty$$

ZÁVĚR: diverguje (do $+\infty$)