CATEGORIES OF MODULES AND HOMOLOGICAL ALGEBRA

EXERCISE 1

Instructions: Please work alone. Try to answer all questions. Questions indicated with (*) are more difficult. It is highly recommended that you type your solutions in LaTeX. If you don't know LaTeX, now is the perfect time to learn it. I am happy to help. Please submit your homework by 11.3.2020 to the email address `catmodules2019@gmail.com`

(1) Let R be a ring, and let $\{E_k\}_{k\in K}$ be a family of injective R-modules. Show that
$$\prod_{k\in K} E_k$$
is an injective R-module. Deduce that if $E_1,\dots,E_n$ are finitely many injective R-modules, then $\oplus_{k=1}^{n} E_k$ is an injective R-module.

(2) (*) Give an example of a ring R, and a collection of projective R-modules $\{P_k\}_{k\in K}$, such that the R-module
$$\prod_{k\in K} P_k$$
is not projective.

(3) Let R be a principal ideal domain. Show that any injective R-module is divisible.

(4) Let R be a ring. Recall that a short exact sequence of left R-modules
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$
is called split if there exists an R-linear map $h : C \to B$ such that $p \circ h = 1_C$. Prove that a short exact sequence is split if and only if there exists an R-linear map $j : B \to A$, such that $j \circ i = 1_A$.

(5) Use (4) to show that an R-module E is injective if and only if every short exact sequence of the form
$$0 \to E \xrightarrow{i} B \xrightarrow{p} C \to 0$$
is split.

(6) Let R, S be rings.

(a) Given an $R-S$ bimodule M and a left R-module N, verify that $\mathrm{Hom}_R(M,N)$ is a left S-module with structure given by
$$(s\cdot f)(m) = f(m\cdot s).$$

(b) Given an $R-S$ bimodule M and a right S-module N, verify that $\mathrm{Hom}_R(M,N)$ is a right R-module with structure given by
$$(f\cdot r)(m) = f(r\cdot m).$$

(c) Define the category of $R-S$-bimoudles $\mathrm{BiMod}_{R,S}$, and check that given a fixed left R-module N, the operation $\mathrm{Hom}_R(-,N)$ defines a functor
$$\mathrm{BiMod}_{R,S} \to \mathrm{Mod}(S)$$