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Microstructure in ferromagnetics and its steady-state and evolution models

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Abstract: This contribution surveys various models in ferromagnetism. The microstructure of magnetization is described on various levels: micro-, meso- and macroscopical. Moreover, for a mesoscopical-level model, thermodynamically consistent extension for anisothermal processes is formulated.

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0. The plan

This survey will present both standard steady-state models used to describe steady-state of *ferromagnetic* rigid bodies as well as some evolution models. Typical phenomenon that makes the modelling rather nontrivial is the *microstructure* of magnetization typically occurring in (even single crystals of) ferromagnets. The steady-state models are based on the *energy minimization* principle in the framework of continuum level. Even on such a level, one can distinguish three levels depending on the resolution with which the microstructure is described, namely a microscopical, a mesoscopical, and (rather for completeness we will also mention) a macroscopical ones; the first level does not mean an atomic level, however. Although minimization of total magnetic energy implicitly expect zero activation and inevitably results to zero hysteresis, which has limited application to magnetically soft materials only, it is illustrative to start the survey with such sort of models. The change of magnetization is, however, an *activated process* yielding a *rate-independent hysteresis*. This will be reflected in evolution models extending naturally the corresponding steady-state models. Besides, an anisothermal extension of the evolution

mesoscopical model will be presented in Section 3. The plan is summarized in the following table:

Level: / Type:	micro	meso	macro
steady-state	Sect.1.1	Sect.1.2	Sect.1.3
evolution	Sect.2.1	Sect.2.2	Sect.2.3
with temperature	—	Sect.3	—

Table 1: The plan of the paper.

Let us emphasize that, while mutual relations between the presented steady-state models are well established, the same cannot be said about evolution models. This is certainly a very challenging question, as well as relations between steady-state models and the evolution ones, which is connected with asymptotical behaviour of the latter ones.

1. Steady-state models in micromagnetism

A configuration of a rigid ferromagnetic body occupying a bounded domain $\Omega \subset \mathbb{R}^3$ can be described, from an electromagnetic viewpoint, by a magnetization $m : \Omega \to \mathbb{R}^3$ depending on a position $x \in \Omega$ and having a given temperature-dependent magnitude

$$|m(x)| = M_{\rm s} = M_{\rm s}(\theta) \quad \text{for almost all } x \in \Omega \ ; \tag{1.1}$$

with $M_{\rm s}$ the so-called *saturation magnetization*; $M_{\rm s}(\theta) > 0$ for $\theta < \theta_{\rm C}$ while $M_{\rm s}(\theta) = 0$ for $\theta \geq \theta_{\rm C}$, the so-called Curie point.

Except Section 3, we will treat the *isothermal case* with the temperature θ fixed below the Curie point.

1.1 Microscopical level.

On microscopical level, the magnetic *Gibbs energy* consists of four parts, namely an *anisotropy energy* $\int_\Omega \varphi(m(x))\,{\rm d}x$ with a density φ which is supposed to be an even nonnegative function depending on material properties and exhibiting crystallographic symmetry, an *exchange energy* $\varepsilon \int_\Omega |\nabla m(x)|^2 {\rm d}x$ having a quantum-theoretical origin ($\varepsilon > 0$ is very small from a macroscopical viewpoint), and eventually a *magnetostatic energy* $\frac{1}{2}\int_{\mathbb{R}^3} |h_{\rm dem}(x)|^2 {\rm d}x$ with the self-induced *demagnetizing field* $h_{\rm dem}$ governed by the equations

$$\nabla \times h_{\rm dem} = 0 \quad \& \quad {\rm div}(h_{\rm dem} + \chi_\Omega m) = 0 \tag{1.2}$$

arising from Maxwell equations after a lot of simplifications, where $\chi_\Omega : \mathbb{R}^3 \to \{0,1\}$ denotes the characteristic function of Ω. The former relation enables us to

introduce a potential u such that $h_{\rm dem} = \nabla u$, so that the magnetostatic energy is $\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(x)|^2 {\rm d}x$ with u related with m through ${\rm div}(\nabla u - m\chi_\Omega) = 0$. Eventually, the fourth part is an *interaction energy* $\int_\Omega h(x)\cdot m(x)\,{\rm d}x$ involving the outer magnetic field h.

A generally accepted model for steady-state configuration is due to Landau and Lifshitz [25, 26] (see e.g. Brown [4, 5, 6] or Hubert and Schäfer [14]), relying on a minimum-of-Gibbs'-energy principle, i.e.

$$
(1.3)\quad \begin{cases} \text{minimize} & E_\varepsilon(m,u) - \displaystyle\int_\Omega h\cdot m\,{\rm d}x & \text{(Gibbs' energy)}\\ \text{where} & E_\varepsilon(m,u) := \displaystyle\int_\Omega \varphi(m) + \varepsilon|\nabla m|^2\,{\rm d}x & \\ & \qquad\qquad + \frac{1}{2}\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\,{\rm d}x, & \text{(Helmholtz' energy)}\\ \text{subject to} & |m| = M_{\rm s} \quad \text{on } \Omega\,, & \text{(magnetization constraint)}\\ & {\rm div}(\nabla u - \chi_\Omega m) = 0 \ \text{ on } \mathbb{R}^3, & \text{(rest of Maxwell's equations)}\\ & m\in L^\infty(\Omega;\mathbb{R}^3),\quad u\in W^{1,2}(\mathbb{R}^3), & \end{cases}
$$

where we used standard notation $L^\infty(\Omega;\mathbb{R}^3)$ for the Lebesgue space of measurable, essentially bounded functions $\Omega\to\mathbb{R}^3$ and $W^{1,2}(\mathbb{R}^3)$ for Sobolev space of functions $u:\mathbb{R}^3\to\mathbb{R}$ belonging together with their distributional derivative to the space of squared integrable functions, i.e. $u\in L^2(\mathbb{R}^3)$ and $\nabla u\in L^2(\mathbb{R}^3;\mathbb{R}^3)$. The part E_ε plays the role of the *Helmholtz free energy*.

This problem is always nonconvex because of the constraint $|m| = M_{\rm s}$. Moreover, we assume that φ attains minimum at several point $\{s_\alpha\}\subset S_{M_{\rm s}} := \{s\in\mathbb{R}^3;\ |s| = M_{\rm s}\}$; each s_α determines a *direction of easy magnetization*. Typical examples are two minimizers s_α for uni-axial magnets and 6 or 8 for cubic magnets. The exchange energy guarantees that the problem (1.3) has a (possibly not unique) solution $(m_\varepsilon, u_\varepsilon)$. However, for ε small, m_ε will typically exhibit fast spatial oscillations, so-called *fine structure*.

Such model has been recently investigated, e.g., by Choksi and Kohn [7], DeSimone [9], DeSimone, Kohn, Müller, Otto and Schäfer [10], James and Müller [16], James and Kinderlehrer [15], Pedregal [34, 35], Rogers [40], Tartar [44] and also in [23, 41].

1.2 Mesoscopical level.

Since $\varepsilon > 0$ is very small, it is natural to consider it only as a singular perturbation and to investigate behaviour when $\varepsilon\to 0$. It leads to a so-called *relaxed problem* (1.4) involving a so-called *Young measure* ν, i.e. a probability measure ν_x on $S_{M_{\rm s}}$ parameterized (in a measurable way) by $x\in\Omega$, which describe the relevant "mesoscopical" character of the fine structure of m. At a given "macroscopical"

point $x \in \Omega$, ν_x represents a *volume fraction* regarding to particular magnetizations on S_{M_s}. It can be proved [9, 34] that the limit configuration $q \equiv (\nu, u)$ solves the following minimization problem involving "mesoscopical" Gibbs' and Helmholtz' energies:

$$\left\{\begin{array}{lll} \text{minimize} & E(\nu,u) - \displaystyle\int_\Omega h \cdot (\mathrm{id} \bullet \nu)\,\mathrm{d}x, & \text{(mesoscopical Gibbs' energy)} \\ \text{where} & E(\nu,u) := \displaystyle\int_\Omega \varphi \bullet \nu\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x, & \text{(Helmholtz' energy)} \\ \text{subject to} & \mathrm{div}\Big(\nabla u - \chi_\Omega(\mathrm{id} \bullet \nu)\Big) = 0 \quad \text{on } \mathbb{R}^3, & \\ & \nu \in \mathcal{Y}(\Omega; S_{M_s}), \quad u \in W^{1,2}(\mathbb{R}^3), & \end{array}\right. \tag{1.4}$$

where we abbreviated $[f \bullet \nu](x) := \int_{\mathbb{R}^3} f(m)\nu_x(\mathrm{d}m)$ and $\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity. Here, $\mathcal{Y}(\Omega; S_{M_s}) \subset L^\infty_{\mathrm{w}}(\Omega; \mathrm{rca}(S_{M_s})) \cong L^1(\Omega; C(S_{M_s}))^*$ denotes the set of all Young measures, i.e. all weakly measurable essentially bounded mappings $x \mapsto \nu_x : \Omega \to \mathrm{rca}(S_{M_s}) \cong C(S_{M_s})^*$ such that ν_x is a probability Radon measure on S_{M_s} for a.a. $x \in \Omega$.

Alternatively, the relaxed problem (1.4) can be viewed simply as a continuous extension of the original problem (1.3) considered for $\varepsilon = 0$. Let us note that the problem (1.4) has a convex structure. Therefore, its solution (ν, u) can be fully characterized by the first-order optimality conditions which read here as:

$$E'(\nu,u) + N_Q(\nu,u) \ni F := (h \otimes \mathrm{id}, 0), \tag{1.5}$$

where E' denotes the Gâteaux differential of E, and $N_Q(\nu,u)$ is the *normal cone* to the admissible set

$$Q := \Big\{(\nu,u) \in \mathcal{Y}(\Omega; S_{M_s}) \times W^{1,2}(\mathbb{R}^3);\ \mathrm{div}(\nabla u - \chi_\Omega(\mathrm{id} \bullet \nu)) = 0\Big\} \tag{1.6}$$

at the point (ν, u), and eventually the tensorial product $h \otimes \mathrm{id} \in L^1(\Omega; C(S_{M_s}))$ is defined naturally by $[h \otimes \mathrm{id}](x,m) := \sum_{i=1}^n h_i(x) m_i$. Let us remark that (1.5) leads to a certain Weierstraß maximum principle, cf. [23] for details. This also enables efficient numerical solution, cf. [21, 22]. Typically, ν is nontrivial (i.e. $\{\nu_x\}_{x\in\Omega}$ are not Dirac measures for a.a. $x \in \Omega$) though sometimes the opposite case may occur, cf. Dacorogna and Fonseca [8].

1.3 Macroscopical level.

Let us also remark that $\mathrm{id} \bullet \nu$ appearing in (1.4) represents a *macroscopical magnetization*, let us denote it by M. It was shown by DeSimone [9] that M solves the "coarsely" relaxed problem involving a convexification of the energy φ augmented

by the indicator function $\delta_{S_{M_\mathrm{s}}}$ of S_{M_s}, i.e.

$$\text{(1.7)}\qquad \begin{cases} \text{minimize} & \bar{E}(M,u) := \displaystyle\int_\Omega \left[\varphi + \delta_{S_{M_\mathrm{s}}}\right]^{**}(M(x)) - h(x)\cdot M(x)\,\mathrm{d}x \\ & \qquad\qquad + \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3} |\nabla u(x)|^2\,\mathrm{d}x, \\ \text{subject to} & \operatorname{div}(\nabla u - \chi_\Omega M) = 0 \ \text{ on } \mathbb{R}^3, \\ & M \in L^\infty(\Omega;\mathbb{R}^3)\ , \quad u \in W^{1,2}(\mathbb{R}^3)\ , \end{cases}$$

where

$$\text{(1.8)}\qquad \delta_{S_{M_\mathrm{s}}}(s) = \begin{cases} 0 & \text{if } |s| = M_\mathrm{s}, \\ +\infty & \text{otherwise,} \end{cases}$$

and $[\,\cdot\,]^{**}$ denotes the convex envelope. The relation between a solution (ν, u) to (1.4) and a solution (M, u) to (1.7) is

$$\text{(1.9)}\qquad M(x) = \mathrm{id}\bullet\nu := \int_{S_{M_\mathrm{s}}} m\,\nu_x(\mathrm{d}m).$$

However, we want to emphasize that the macroscopical viewpoint already neglects the information about microstructure which will certainly be further essential to determine an evolution.

2. Evolution models

When the outer magnetic field h varies in time, the configuration (m, u) (see (1.3)) or (ν, u) (see (1.4)) eventually may start to evolve, too. Experimentally it has repeatedly been proved that this evolution is an activated process accompanied by dissipation that leads, except very-high-frequency ranges, to a very typical rate-independent hysteretic response of the magnet. There are models on each mentioned level reflecting this phenomenon.

2.1 Microscopical level.

A standard evolution model is the Gilbert-Landau-Lifshitz one [12, 25], governed by the equation

$$\text{(2.1)}\qquad \frac{\partial m}{\partial t} = \lambda_1 m \times h_{\text{eff}} - \lambda_2 m \times (m \times h_{\text{eff}}), \quad h_{\text{eff}} := h - \varphi'(m) + \varepsilon\Delta m - \frac{1}{2}\nabla u,$$

while u is again determined from $\operatorname{div}(\nabla u - \chi_\Omega m) = 0$. Here φ' stands for a derivative of φ, assumed as defined also outside S_{M_s}. The balance of magnetic energy E_ε (see (1.3)) can be obtained by multiplying (2.1) by the "effective magnetic field" h_{eff} and integrating over Ω, yielding

$$\text{(2.2)}\qquad \frac{\mathrm{d}E_\varepsilon(m,u)}{\mathrm{d}t} = -\int_\Omega h_{\text{eff}}\cdot\frac{\partial m}{\partial t}\,\mathrm{d}x = -\lambda_2\int_\Omega |m\times h_{\text{eff}}|^2\mathrm{d}x \le 0,$$

which expresses Clausius-Duhem's inequality. In particular, the "precession" λ_1-term does not dissipate energy, while λ_2-term determines, roughly speaking, phenomenologically a "viscous" damping of the process.

The multiwell structure of φ (restricted on S_{M_s}) together with the λ_2-term may cause a nearly rate-independent hysteretic response, which is typical for ferromagnetical materials; cf. Blatensperger and Helman [2] for pursuit of this idea on microscopical level. The width of the hysteresis loop in the m/h-diagram, which determines dissipated energy, can thus be indirectly controlled by a shape of φ. This is, however, very implicit and that is why the applications of the model (2.1) are generally accepted rather with limitations, though it has often been used (see e.g. [27, 39, 43]), even for commercial codes (see [33]).

One can be inspired by models used in classical mechanics for plasticity and/or *dry friction* phenomena to augment the dissipation mechanism in (2.1). Indeed, Visintin [46] augmented $h_{\rm eff}$ in (2.1) by a set-valued term $L\partial|\cdot|$, $L>0$, so that

$$h_{\rm eff} \in h - \varphi'(m) + \varepsilon\Delta m - \frac{1}{2}\nabla u + L\omega(\frac{\partial m}{\partial t}), \tag{2.3}$$

$$\omega(z) = \begin{cases} z/|z| & \text{if } z \neq 0, \\ \{z\in\mathbb{R}^3;\ |z|\le 1\} & \text{if } z = 0. \end{cases}$$

This leads to the additional dissipation term

$$L\int_\Omega \left|\frac{\partial m}{\partial t}\right| {\rm d}x \tag{2.4}$$

in the energy balance (2.2). This dissipation mechanism is rate-independent which may cause that the width of the h/m-hysteresis loop can be determined explicitly by the phenomenological parameter L, assuming that influence of the viscous damping through λ_2-term is suppressed for sufficiently slow loading processes $h = h(t)$. For such sort of models but in multi-well elasticity, this effect was computationally tested in [36] in the context of shape-memory alloys.

For extension of this model considering full Maxwell system instead of (1.2) we refer to Visintin [47], while for other dry-friction type models we refer to Bergqvist [1]. For simulation of a hysteresis in a pure quasistationar model see Kinderlehrer and Ma [19].

2.2 Mesoscopical level.

Certain disadvantage of the models from Sect.2.1 is that they can describe rather micro-scale (at most 10^{-3}m scale where the exchange energy dominates) than engineering-scale situations. This can be overcome by a mesoscopical model.

If we want to define the evolution $t \mapsto q(t) \equiv (\nu(t), u(t))$ with $\nu(t)$ describing the mesoscopical microstructure at a given time instance t like in (1.4), we must also postulate the generalized *impulse* $\dot q \equiv (\dot\nu, \dot u)$ with the dot indicating the time

derivative, for which we need some geometric structure. This is, in fact, a matter of a certain choice: it appears fruitful to take the convex geometry of Q induced from the linear space $L^1(\Omega;C(S_{M_s}))^* \times W^{1,2}(\mathbb{R}^3)$, which will yield the desired response as indicated by the energy balance (2.7)–(2.9) and demonstrated computationally in [24, 42].

The dissipation (2.4) could be implemented on the mesoscopical level by the *dissipation function* R (i.e. the potential of generalized dissipative force) in the form $R(\dot\nu,\dot u)=L\int_\Omega|\mathrm{id}\bullet\dot\nu|\,\mathrm{d}x$. For *uni-axial magnets* (oriented in x_3-direction) it rather occurs that only the x_3-component of $\mathrm{id}\bullet\dot\nu$ dissipates. Therefore, for K an *anisotropy parameter*, the data φ and R can be considered as

$$\varphi(m)=\varphi(m_1,m_2,m_3)=K(m_1^2+m_2^2)\ , \tag{2.5a}$$

$$R(\dot\nu,\dot u)=\int_\Omega|\lambda\bullet\dot\nu|\,\mathrm{d}x\quad\text{with}\quad\lambda(m)=H_{\mathrm{c}}m_3; \tag{2.5b}$$

recall that we defined $[\lambda\bullet\dot\nu](x):=\int_{\mathbb{R}^3}\lambda(m)\dot\nu_x(\mathrm{d}m)$. Now we wrote H_{c} instead of L, the parameter H_{c} having the definite meaning of the *coercive field*, cf. Figure 1 below. In view of (1.9), (2.5b) can also be written as $R(\dot\nu,\dot u)=\int_\Omega|\dot M_3|\,\mathrm{d}x$.

Likewise the plasticity models in metals and shape-memory alloys, the desired dissipation/hysteretic effects can be achieved by the evolution $t\mapsto q(t)$ governed by the following first-order evolution inclusion (see [42], cf. also Eve, Reddy, and Rockafellar [11], Krejčí [20], or Mielke, Theil and Levitas [31, 32] for general investigations):

$$\partial R(\frac{\mathrm{d}q}{\mathrm{d}t})+E'(q)+N_Q(q)\ni F(t)\ ,\qquad q(0)=q_0, \tag{2.6}$$

where $\partial R(\dot q)=\{z\in L^1(\Omega;C(S_{M_s}))\times W^{-1,2}(\mathbb{R}^3);\ \forall\tilde q\in L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(S_{M_s}))\times W^{1,2}(\mathbb{R}^3):\ R(\tilde q)\ge R(\dot q)+\langle z,\tilde q-\dot q\rangle\}$ denotes the subdifferential of R at $\dot q$. The set-valued mapping ∂R is monotone and even so-called *maximal responsive*, see [11] for a deep investigation. Besides, $q_0\equiv(\nu_0,u_0)$ is the initial configuration; in fact, only the momenta of q_0 involved in R are relevant, which means here that only $\lambda\bullet\nu_0$ is to be set up for, i.e. one must set up the *initial volume fraction* of the poles.

A justification of the model (2.6) is the desired energy balance which can be then obtained, at least formally, by testing the inclusion (2.6) by $\frac{\mathrm{d}}{\mathrm{d}t}q$ and integrating over a time interval, say $[0,T]$. This gives, for any $z\in N_Q(q)$, that

$$\begin{aligned}0\ &=\ -\int_0^T\langle z,\frac{\mathrm{d}q}{\mathrm{d}t}\rangle\,\mathrm{d}t=\int_0^T\left\langle\partial R(\frac{\mathrm{d}q}{\mathrm{d}t})+E'(q)-F(t),\frac{\mathrm{d}q}{\mathrm{d}t}\right\rangle\,\mathrm{d}t\\&=\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}E(q)+\xi(\frac{\mathrm{d}q}{\mathrm{d}t})-\langle F(t),\frac{\mathrm{d}q}{\mathrm{d}t}\rangle\right)\mathrm{d}t\end{aligned} \tag{2.7}$$

$$= \underbrace{E(q(T))}_{\substack{\text{final}\\ \text{energy}}} + \underbrace{\int_0^T \xi(\frac{\mathrm{d}q}{\mathrm{d}t})\,\mathrm{d}t}_{\substack{\text{dissipated}\\ \text{energy}}} - \underbrace{\int_0^T \langle F(t), \frac{\mathrm{d}q}{\mathrm{d}t}\rangle\,\mathrm{d}t}_{\substack{\text{work made by}\\ \text{external field } h}} - \underbrace{E(q_0)}_{\substack{\text{initial}\\ \text{energy}}}$$

where ξ denotes the *rate of dissipation* given here by

$$\xi(\frac{\mathrm{d}q}{\mathrm{d}t}) = \langle z, \frac{\mathrm{d}q}{\mathrm{d}t}\rangle = \int_\Omega \left|\lambda \bullet \frac{\mathrm{d}\nu}{\mathrm{d}t}\right| \mathrm{d}x \qquad \forall z \in \partial R(\frac{\mathrm{d}q}{\mathrm{d}t})\ . \tag{2.8}$$

One can also notice that (2.7) expresses, in particular, the *Clausius-Duhem inequality* $\frac{\mathrm{d}}{\mathrm{d}t}E(q) - \langle F, \frac{\mathrm{d}}{\mathrm{d}t}q\rangle \le 0$, cf. Brown [5, Sect.3.2] or also Bergqvist [1]. Especially, having in mind (2.5), the bilance (2.7) turns into

$$E(\nu(T), u(T)) + \int_\Omega \left(\operatorname*{Var}_{t\in[0,T]} \lambda \bullet \nu \right) \mathrm{d}x = \int_0^T \int_\Omega h \cdot (\mathrm{id} \bullet \nu)\,\mathrm{d}x\mathrm{d}t + E(\nu_0, u_0) \tag{2.9}$$

where "Var" denotes the total variation over the time interval indicated. This is just the desired effect: the energy dissipated in a "macroscopical infinitesimally small volume" $\mathrm{d}x$ counts how many times the pole transformed within the time interval $[0, T]$ in $\mathrm{d}x$.

Moreover, as the subdifferential ∂R is assumed maximal responsive, by [11, Lemma 4.1(c,d)] the latter inclusion in (2.8) is equivalent to

$$\langle \frac{\mathrm{d}q}{\mathrm{d}t}, z\rangle = \max_{\tilde{z}\in\mathfrak{C}} \langle \frac{\mathrm{d}q}{\mathrm{d}t}, \tilde{z}\rangle \tag{2.10}$$

where the convex set $\mathfrak{C} = \partial R(0) = \{z;\ \forall q :\ \langle z, q\rangle \le R(q)\}$ determines the region of nondissipative (i.e. nonhysteretic) response. The relation (2.10) is just what is called (in plasticity Hill's [13]) *maximum-dissipation principle*, and expresses the rule that the rate of change of the configuration q is normal to $\mathfrak{C}$ at z where, from (2.6), one has $z + E'(q) + N_Q(q) \ni F(t)$. For a connection with a (vector) *play operator* routinely used in hysteresis theory see Brokate and Sprekels [3], Krejčí [20] or Visintin [45]. By analyzing the abstract principle (2.10) for the special case (2.5b), one can identify the point-wise explicit activation rule that triggers the magnetization evolution process:

$$\frac{\mathrm{d}M_3}{\mathrm{d}t} \begin{cases} = 0 & \Longleftarrow \quad -H_\mathrm{c} < \mathfrak{H} < H_\mathrm{c}, \\ > 0 & \Longrightarrow \quad \mathfrak{H} = H_\mathrm{c}, \\ < 0 & \Longrightarrow \quad \mathfrak{H} = -H_\mathrm{c}. \end{cases} \tag{2.11}$$

cf. [42, Formula (5.13)]. Moreover, one can see that the scalar function $\mathfrak{H} = \mathfrak{H}(x,t)$ appearing in (2.11), which plays a role of an *effective field* activating the magnetization process, satisfies $\mathfrak{H}(x,t) \in H_c\mathrm{sign}(M_3)$ where $\mathrm{sign}(M_3)$ denotes the set-valued function being equal to (resp. -1) for M_3 positive (resp. negative) and to the interval [-1,1] for $M_3 = 0$.

The existence of a weak solution to the inclusion (2.6) involving, however, a certain regularization of E has been proved in [42].

2.3 Macroscopical level.

For completeness, let us mention phenomenological models that do not involve explicitly any microstructure. Such models are quite popular among engineers and physicists, too. Classical approaches are Rayleigh's modification [38] of Prandtl and Ishlinskiĭ model or Preisach's model [37] (see also Jiles [17], Mayergoyz [30], or Visintin [45]). This makes possible to involve even a continuum of activation thresholds. Another one is, e.g., due to Jiles and Atherton [18].

Besides, the one-threshold dry-friction idea like (2.11) has been used in a model based on macroscopical magnetization M from (1.9) by Visintin [47].

3. Thermodynamical evolution on mesoscopical level

We already mentioned dependence of (1.4) on temperature θ through dependence of $S_{M_{\rm s}} = S_{M_{\rm s}(\theta)}$, cf. (1.1). So far, we considered isothermal processes, assuming that the dissipated magnetic energy does not influence temperature θ.

Of course, a more realistic model must consider a temperature field $\theta = \theta(x,t)$ to be coupled with the evolution of magnetic field. The model (2.6) clearly specified the amount of dissipated magnetic energy and, of course, this energy eventually transforms to chaotic vibration of atoms, i.e. to heat. This heat energy may increase the temperature θ. Let us develop formally a thermodynamically consistent theory based on the mesoscopical level model (2.6).

The simplest possibility is to consider a constant *specific heat* $c > 0$, which contributes to the specific Helmholtz *free energy* ψ by a term $-c\theta\ln(\theta)$, so that it gets the form

$$\psi(\nu,u,\theta) = \chi_\Omega\Big(\varphi\bullet\nu + \delta_{M_{\rm s}(\theta)}(\nu) - c\theta\ln(\theta)\Big) + \frac{1}{2}|\nabla u|^2, \tag{3.1}$$

$$\text{where } \delta_{M_{\rm s}(\theta)}(\nu) := \begin{cases} 0 & \text{if } \operatorname{supp}(\nu_x) \in S_{M_{\rm s}(\theta(x))} \\ & \qquad \text{for a.a. } x\in\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Again, $u \in W^{1,2}(\mathbb{R}^3)$ is related with ν by $\operatorname{div}(\nabla u - \chi_\Omega(\mathrm{id}\bullet\nu)) = 0$.

This may cause, as a side effect, a temperature dependence of the dissipated energy as well as of the anisotropy. For example, in the case (2.5), the dependence of the h/m-diagramme is on Figure 1 for a model 3D axisymetrical case with Ω a cylinder with the axis x_3 (i.e. oriented vertically, cf. Figure 2) as well as the easy-magnetization axis and the external magnetic field $h = h(t)$ and θ constant over Ω, playing a role of a parameter:

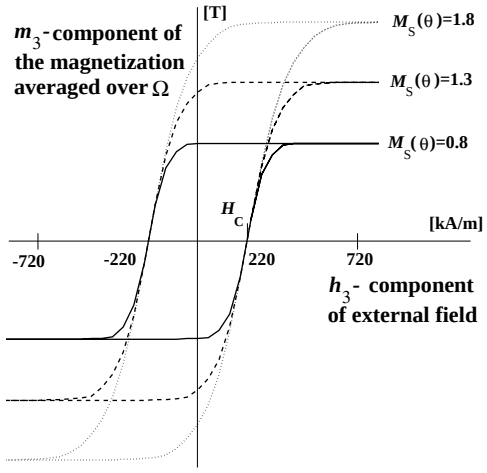

Fig.1: h/m-diagramme for a cylindrical uni-axial magnet.

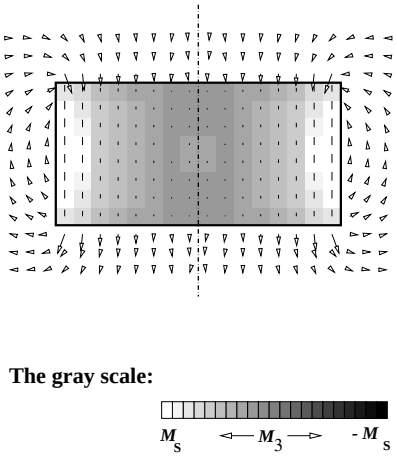

Fig.2: The cross-section of Ω with one snapshot of the macroscopical magnetization M_3 inside Ω and the demagnetizing field $h_{\rm dem}$ around Ω.

The varying gray levels on Figure 2 indicate inhomogeneity of macroscopical magnetization inside the bulk magnet Ω, which causes the hysteresis loops on Figure 1 to be curved. The dissipated energy (per unit volume) within the transformation of one pole $m = (\pm M_{\rm s}(\theta), 0, 0)$ into the other one $(\mp M_{\rm s}(\theta), 0, 0)$ is then $2H_{\rm c}M_{\rm s}(\theta)$, so that the area of the hysteresis loop in h/m-diagram is $4H_{\rm c}M_{\rm s}(\theta)$.

To pursue the standard thermodynamical procedure, it seems more suitable to transform ψ by introducing a *normalized magnetization* μ supported on the unit sphere $S_1 \subset \mathbb{R}^3$, i.e. $\mu \in \mathcal{Y}(\Omega; S_1)$, related with ν by

$$\nu = T^*_{M_{\rm s}(\theta)}\mu, \quad \text{with } T^*_{M_{\rm s}(\theta)} = \big(T_{M_{\rm s}(\theta)}\big)^*, \quad \text{where } T_{M_{\rm s}(\theta)}h(x,s) := h(x, M_{\rm s}(\theta(x))s) \tag{3.2}$$

and where $(\cdot)^*$ denotes the adjoint operator. To explain this definition, let us realize that for $\mu = i(\tilde{m})$ one has $\nu = i(m)$ with $m = M_{\rm s}(\theta)\tilde{m}$. Indeed, one has the following simple chain: $\langle \nu, h\rangle = \langle T^*_{M_{\rm s}(\theta)}\mu, h\rangle = \langle T^*_{M_{\rm s}(\theta)} i(\tilde{m}), h\rangle = \langle i(\tilde{m}), T_{M_{\rm s}(\theta)}h\rangle = \int_\Omega h(x, M_{\rm s}(\theta(x))\tilde{m}(x))\,{\rm d}x = \langle i(M_{\rm s}(\theta)\tilde{m}), h\rangle = \langle i(m), h\rangle$. Also, one has $\mathrm{id} \bullet T^*_{M_{\rm s}(\theta)}\mu = M_{\rm s}(\theta)(\mathrm{id} \bullet \mu)$.

Considering the special case (2.5), this gives the transformed specific free energy and dissipation rate

$$\tilde{\psi}(\mu, u, \theta) = \chi_\Omega \Big(M_{\rm s}(\theta)^2 \varphi \bullet \mu + \delta_1(\mu) - c\theta\ln(\theta)\Big) + \frac{1}{2}|\nabla u|^2, \tag{3.3a}$$

$$\tilde{\xi}(\frac{{\rm d}\mu}{{\rm d}t}, \theta) = M_{\rm s}(\theta) \left| \lambda \bullet \frac{{\rm d}\mu}{{\rm d}t} \right|, \tag{3.3b}$$

respectively. Now u and μ are coupled by

$$\operatorname{div}(\nabla u - M_{\rm s}(\theta)\chi_\Omega(\mathrm{id}\bullet\mu)) = 0. \tag{3.4}$$

The equation (2.6) now uses Q depending on θ, which results to the equation for (μ, u):

$$\partial_{(\mu,u)}\tilde R(\frac{\mathrm{d}(\mu,u)}{\mathrm{d}t},\theta) + \tilde\Psi'_{(\mu,u)}(\mu,u,\theta) + N_{\tilde Q(\theta)} \ni \tilde F(t,\theta) \tag{3.5}$$

with $\tilde\Psi(\mu,u,\theta) = \int_{\mathbb{R}^3}\tilde\psi(\mu,u,\theta)\mathrm{d}x$ with $\tilde\psi$ from (3.3a), $\tilde R(\frac{\mathrm{d}}{\mathrm{d}t}(\mu,u),\theta) = \tilde\xi(\frac{\mathrm{d}}{\mathrm{d}t}\mu,\theta)$ with $\tilde\xi$ from (3.3b),

$$\tilde Q(\theta) = \{(\mu,u)\in\mathcal{Y}(\Omega;S_1)\times W^{1,2}(\mathbb{R}^3);\quad (3.4)\text{ holds}\}, \tag{3.6}$$

and $\tilde F(t,\theta) = (M_{\rm s}(\theta)(h(t)\otimes\mathrm{id}),0)$.

Then the standard thermodynamically consistent theory would define a specific entropy by $s = -\partial\tilde\psi/\partial\theta$. Here, however, $\tilde\psi$ depends on θ also through u via the equation (3.4) which has a nonlocal character. Thus we must start with the total free energy $\tilde\Psi(\mu,u,\theta) = \int_{\mathbb{R}^3}\tilde\psi(\mu,u,\theta)\mathrm{d}x$ and seek s as the gradient $-\tilde\Psi'_\theta(\mu,u,\theta)$, i.e. $\int_{\mathbb{R}^3} s\vartheta\,\mathrm{d}x$ equals to the directional derivative $-[\tilde\Psi'_\theta(\mu,u,\theta)](\vartheta)$ of $\tilde\Psi$ for any direction ϑ. One can call s the specific *entropy* and find the nonlocal formula

$$\begin{aligned} s = \chi_\Omega\Big(&-2M'_{\rm s}(\theta)M_{\rm s}(\theta)(\varphi\bullet\mu)\\ &- M'_{\rm s}(\theta)(\mathrm{id}\bullet\mu)\cdot\nabla\Delta^{-1}\operatorname{div}(\chi_\Omega M_{\rm s}(\theta)(\mathrm{id}\bullet\mu)) + c(1+\ln(\theta))\Big) \end{aligned} \tag{3.7}$$

and then, through Gibbs' relation, the specific *internal energy*

$$\begin{aligned} e = \psi+\theta s = \chi_\Omega\Big(&(M_{\rm s}(\theta)^2 - 2\theta M'_{\rm s}(\theta)M_{\rm s}(\theta))(\varphi\bullet\mu)\\ &- \theta M'_{\rm s}(\theta)(\mathrm{id}\bullet\mu)\cdot\nabla\Delta^{-1}\operatorname{div}(\chi_\Omega M_{\rm s}(\theta)(\mathrm{id}\bullet\mu)) + c\theta\Big) + \frac{1}{2}|\nabla u|^2\,. \end{aligned} \tag{3.8}$$

The classical energy balance says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} e(x)\,\mathrm{d}x = \text{power of external forces} = \int_\Omega M_{\rm s}(\theta)h\cdot(\mathrm{id}\bullet\mu)\,\mathrm{d}x. \tag{3.9}$$

Testing (3.5) by $\frac{\mathrm{d}}{\mathrm{d}t}(\mu,u)$ gives, after using (3.8) and $s=-\tilde\Psi'_\theta$ and also (3.9), the following balance:

$$\begin{aligned} 0 &= \Big\langle \tilde\Psi'_{(\mu,u)}(\mu,u,\theta), \frac{\mathrm{d}}{\mathrm{d}t}(\mu,u)\Big\rangle + \tilde R(\frac{\mathrm{d}(\mu,u)}{\mathrm{d}t},\theta) - \Big\langle \tilde F(t,\theta), \frac{\mathrm{d}}{\mathrm{d}t}(\mu,u)\Big\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde\Psi(\mu,u,\theta) - \Big\langle \tilde\Psi'_\theta(\mu,u,\theta), \frac{\mathrm{d}\theta}{\mathrm{d}t}\Big\rangle + \tilde R(\frac{\mathrm{d}(\mu,u)}{\mathrm{d}t},\theta) - \Big\langle \tilde F(t,\theta), \frac{\mathrm{d}}{\mathrm{d}t}(\mu,u)\Big\rangle\\ &= \int_{\mathbb{R}^3}\frac{\partial\tilde\psi}{\partial t}\,\mathrm{d}x + \int_\Omega\Big(s\frac{\partial\theta}{\partial t} + \tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta) - M_{\rm s}(\theta)h\cdot(\mathrm{id}\bullet\mu)\Big)\,\mathrm{d}x \end{aligned} \tag{3.10}$$

$$= \int_{\mathbb{R}^3} \frac{\partial e}{\partial t}\,\mathrm{d}x + \int_\Omega \left(-\theta\frac{\partial s}{\partial t} + \tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta) - M_{\mathrm{s}}(\theta)h\cdot(\mathrm{id}\bullet\mu)\right)\mathrm{d}x$$
$$= \int_\Omega \left(\tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta) - \theta\frac{\partial s}{\partial t}\right)\mathrm{d}x.$$

Furthermore, we consider isotropical heat conduction through Fourier's law, i.e. the heat flux is $-\kappa\nabla\theta$ with κ denoting heat conductivity coefficient of the material. The total energy balance (3.10) then yields the following entropy equation

$$\theta\frac{\partial s}{\partial t} + \mathrm{div}(\kappa\nabla\theta) = \text{dissipation rate} = \tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta)\ . \tag{3.11}$$

Substituting s from (3.7) gives the equation for temperature

$$c\frac{\partial\theta}{\partial t} - \mathrm{div}(\kappa\nabla\theta) = \tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta) - \theta\frac{\partial}{\partial t}\Big(2M_{\mathrm{s}}'(\theta)M_{\mathrm{s}}(\theta)(\varphi\bullet\mu) + M_{\mathrm{s}}'(\theta)(\mathrm{id}\bullet\mu)\cdot\nabla\Delta^{-1}\mathrm{div}(\chi_\Omega M_{\mathrm{s}}(\theta)(\mathrm{id}\bullet\mu))\Big), \tag{3.12}$$

which closes the system through (3.5)–(3.6).

Let us also remark that the system (3.5)–(3.6)–(3.12) is consistent with 2nd thermodynamical law. E.g., assuming the magnet Ω thermally isolated, which corresponds to the Neumann boundary conditions for θ, the *Clausius-Duhem inequality* for the total entropy $\int_\Omega s\,\mathrm{d}x$ can be obtained by multiplying (3.11) by $1/\theta$, integrating it over Ω, and applying Green's formula. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega s\,\mathrm{d}x = \int_\Omega \frac{\tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta) - \mathrm{div}(\kappa\nabla\theta)}{\theta}\,\mathrm{d}x = \int_\Omega \frac{\tilde\xi(\frac{\mathrm{d}\mu}{\mathrm{d}t},\theta)}{\theta} + \kappa\frac{|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x \ge 0 \tag{3.13}$$

valid if $\theta \ge 0$, which is ensured in case the initial condition for (3.12) is non-negative.

Let us emphasize, however, that rigorous mathematical theory of the coupled system (3.5)–(3.6)–(3.12) seems highly nontrivial and is not developed yet.

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