SHARPNESS OF THE ASSUMPTIONS FOR THE REGULARITY OF A HOMEOMORPHISM

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Abstract. The recent result shows that a homeomorphism $f \in W^{1,n-1}_{\rm loc}(\Omega, \mathbb{R}^n)$ of finite distortion satisfies $f^{-1} \in W^{1,1}_{\rm loc}(f(\Omega), \mathbb{R}^n)$. We show that this result is sharp in a sense that the crutial regularity condition $|Df| \in L^{n-1}$ cannot be replaced by $|\operatorname{adj} Df| \in L^1$ or by a requirement that $|Df|$ belongs to some bigger Orlicz space.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a mapping $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^n)$ has finite (outer) distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0 \Rightarrow |Df(x)| = 0$ a.e. Moreover we say that a mapping $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^n)$ has finite inner distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0 \Rightarrow |\operatorname{adj} Df| = 0$ a.e. (for basic properties, examples and applications see e.g. [10]). Here $\operatorname{adj} A$ means an adjugate matrix - see Preliminaries for the definition.

Our aim is to show the sharpness of the following recent result from [1] (see also [6], [7], [14], [11] and [8]):

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\rm loc}(\Omega, \mathbb{R}^n)$ be a homeomorphism of finite inner distortion. Then $f^{-1} \in W^{1,1}_{\rm loc}(f(\Omega), \mathbb{R}^n)$ and f^{-1} is a mapping of finite outer distortion. Moreover*

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_{\Omega} |\operatorname{adj} Df(x)|\, dx. \tag{1.1}$$

This statement is actually claimed in [1] only for mappings of finite outer distortion. However with a very slight modification of the arguments given there (see Section 3 for details) it is possible to show the statement also for a wider class of mappings of finite inner distortion (see also [4]). Also formula (1.1) is not shown there, but it was previously shown under stronger assumptions in [7] and under $W^{1,n-1}$ regularity assumption in [16]. Let us also note that the assumption that f has finite inner distortion is not artificial, because it was shown in [9, Theorem 4] that each homeomorhism such that $f \in W^{1,1}_{\rm loc}$, $J_f \geq 0$ a.e. and $f^{-1} \in W^{1,1}_{\rm loc}$ is necessarily a mapping of finite inner distortion.

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Our aim is to show that assumptions of Theorem 1.1 are sharp in a sense that the crutial regularity condition $|Df| \in L^{n-1}_{\text{loc}}$ cannot be weakened. From the inequality (1.1) one may be tempted to believe that for a conclusion $Df^{-1} \in L^1$ is could be enough to assume that $\operatorname{adj} Df \in L^1$. We show that this is not true:

Example 1.2. *Let $0 < \varepsilon < 1$ and $n \geq 3$. There exist a domain $\Omega \subset \mathbb{R}^n$ and homeomorphism $f \in W^{1,n-1-\varepsilon}(\Omega, \mathbb{R}^n)$ such that $|\operatorname{adj} Df| \in L^1(\Omega)$, pointwise derivative ∇f^{-1} exists a.e. in $f(\Omega)$, but $|\nabla f^{-1}| \notin L^1(f(\Omega))$.*

It is known that for any $n \geq 3$ and $0 < \varepsilon < 1$ there exists homeomorphism $f \in W^{1,n-1-\varepsilon}$ such that $f^{-1} \notin W^{1,1}_{\text{loc}}$ (see [8, Example 3.1] or Example 1.2) and therefore Theorem 1.1 is sharp on a scale of Sobolev spaces. Let us note that for many problems connected with the theory of mapping of finite distortion the optimal regularity of Df is not on the Lebesgue scale, but on some finer Orlicz scale (see [13] and references given there). We show that this is not the case for Theorem 1.1 and no smaller integrability condition of Df is enough.

Example 1.3. *Let $n \geq 3$ and suppose that $g : [0, \infty) \to (0, \infty)$ is a decreasing function such that*

$$\lim_{s\to\infty} g(s) = 0.$$

Then there is a homeomorphism $f \in W^{1,1}(B(0,1); \mathbb{R}^n)$ such that

$$\int_{B(0,1)} |Df(x)|^{n-1} g(|Df(x)|)\, dx < \infty, \tag{1.2}$$

pointwise derivative ∇f^{-1} exists almost everywhere in $f(B(0,1))$, but $|\nabla f^{-1}| \notin L^1_{\text{loc}}(f(B(0,1)))$.

Let us point out that the conclusion of our examples that ∇f^{-1} exists and is not integrable imply that $f^{-1} \notin W^{1,1}$ and even that $f^{-1} \notin BV$.

2. Preliminaries

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathcal{L}_n(A)$.

Given a square matrix $B \in \mathbb{R}^{n\times n}$, we define the norm $|B|$ as the supremum of $|Bx|$ over all vectors x of unit euclidean norm. The adjugate $\operatorname{adj} B$ of a regular matrix B is defined by the formula

$$B \operatorname{adj} B = I \det B, \tag{2.1}$$

where $\det B$ denotes the determinant of B and I is the identity matrix. The operator adj is then continuously extended to $\mathbb{R}^{n\times n}$.

2.1. **Differentiability of radial functions.** By $\|x\|$ we denote the norm of $x \in \mathbb{R}^n$, in fact we use either euclidean norm or maximum norm $\|x - y\| = \max\{|x_i - y_i| : \ i = 1, \dots, n\}$. The following lemma can be verified by an elementary calculation for the euclidean norm. The maximum norm can be obtained from the euclidean norm by the bilipschitz change of variables and therefore it is easy to check that the formulas hold also for this norm.

Lemma 2.1. *Let $\rho : (0,\infty) \to (0,\infty)$ be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\rho(\|x\|), \quad x \neq 0$$

we have for almost every x

$$Df(x) \sim \max\Big\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\Big\},\ J_f(x) \sim \rho'(\|x\|)\Big(\frac{\rho(\|x\|)}{\|x\|}\Big)^{n-1}$$

$$\text{and } |\operatorname{adj} Df(x)| \sim \max\Big\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\Big\}\Big(\frac{\rho(\|x\|)}{\|x\|}\Big)^{n-2}.$$

2.2. **Area formula.** We say that a mapping $f : \Omega \to \mathbb{R}^n$ satisfies the *Lusin condition* (N) if the implication $|S| = 0 \implies |f(S)| = 0$ holds for any measurable set $S \subset \Omega$.

Let $f \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^n)$ be a homeomorphism and let η be a non-negative Borel-measurable function on $\mathbb{R}^n$. Without any additional assumption we have

$$\int_\Omega \eta(f(x))|J_f(x)|dx \leq \int_{\mathbb{R}^n} \eta(y)dy. \tag{2.2}$$

Moreover there exists a set $\Omega' \subset \Omega$ of full measure such that the area formula holds for f on Ω':

$$\int_{\Omega'} \eta(f(x))|J_f(x)|dx = \int_{f(\Omega')} \eta(y)dy \tag{2.3}$$

Also, the area formula holds on each set on which the Luzin condition (N) is satisfied. This follows from the area formula for Lipschitz mappings, from the a.e. approximate differentiability of f [3, Theorem 3.1.4], and a general property of a.e. approximately differentiable functions [3, Theorem 3.1.8], namely that Ω can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz continuous.

3. Finite inner distortion

The following lemma from [1, Lemma 4.3] contains the main ingredient for the proof of Theorem 1.1.

Lemma 3.1. *Let $f \in W^{1,n-1}_{\rm loc}(\Omega,\mathbb{R}^n)$ be a homeomorphism. Then*

$$\int_B |f^{-1}(y) - c|\, dy \leq C r_0 \int_{f^{-1}(B)} |\operatorname{adj} Df(x)|\, dx, \tag{3.1}$$

for each ball $B = B(y_0, r_0) \subset f(\Omega)$, where

$$c = \fint_B f^{-1}(y)\, dy$$

and $C = C(n)$.

The following theorem was shown in [1, Theorem 4.5].

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism such that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and $J_f \geq 0$ a.e. Then f^{-1} is a mapping of finite outer distortion.*

In order to prove the equality (1.1) we will need the following technical lemma from [4, Lemma 2.1].

Lemma 3.3. *Let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism such that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$. Set*

$$E = \big\{ y \in f(\Omega) : f^{-1} \text{ is approximatively differentiable at } y \text{ and } |J_{f^{-1}}(y)| > 0 \big\}.$$

Then there exists a Borel set $A \subset E$ such that $|E \setminus A| = 0$,

$$f^{-1}(A) \subset \tilde{E} := \{ x \in \Omega : f \text{ is approximatively differentiable at } x \text{ and } |J_f(x)| > 0 \}$$

$$\text{and } Df^{-1}(y) = \big[Df(f^{-1}(y)) \big]^{-1} \quad \text{for every } y \in A. \tag{3.2}$$

Moreover $|\tilde{E} \setminus f^{-1}(A)| = 0$.

Proof. It is enough to show that $|\tilde{E} \setminus f^{-1}(A)| = 0$, because everything else is stated and shown in [4, Lemma 2.1]. Suppose for contradiction that there is a Borel set $G \subset \tilde{E} \setminus f^{-1}(A)$ such that $|G| > 0$. Without loss of generality we can also suppose that (2.3) holds for G (i.e. $G \subset \Omega'$) and thus

$$\int_G J_f(x)\, dx = \int_{\mathbb{R}^n} \chi_{f(G)}(y)\, dy = |f(G)|.$$

Since $J_f > 0$ on G we obtain that $|f(G)| > 0$. We know that the area formula holds for f^{-1} on a Borel subset $M \subset f(G)$ of full measure. From

$$\int_{f^{-1}(M)} J_f(x)\, dx = |M| > 0$$

we obtain that $|f^{-1}(M)| > 0$. Therefore we can use area formula for f^{-1} to conclude

$$\int_{f(G) \cap M} |J_{f^{-1}}(y)|\, dy = |G \cap f^{-1}(M)| > 0.$$

It follows that $J_{f^{-1}} > 0$ on a subset of $f(G)$ of positive measure. Clearly f^{-1} is approximatively differentiable a.e. on $f(G)$ and therefore $f(G) \cap A \neq \emptyset$ gives us a contradiction. $\square$

Proof of Theorem 1.1. We claim that there is a function $g \in L^1_{\mathrm{loc}}(f(\Omega))$ such that

$$\int_{f^{-1}(B)} |\operatorname{adj} Df| = \int_B g. \tag{3.3}$$

This and Lemma 3.1 imply that the pair f, g satisfies a 1-Poincaré inequality in $f(\Omega)$. From [2, Theorem 9] we then deduce that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.

There is a set $\Omega' \subset \Omega$ of full measure such that the area formula (2.2) holds for f on Ω'. We define a function $g\colon f(\Omega) \to \mathbb{R}$ by setting

$$g(f(x)) = \begin{cases} \frac{|\operatorname{adj} Df(x)|}{J_f(x)} & \text{if } x \in \Omega' \text{ and } J_f(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is a mapping of finite inner distortion, we have

$$|\operatorname{adj} Df(x)| = g(f(x))\, J_f(x) \text{ a.e. in } \Omega.$$

Hence for every $A \subset f(\Omega)$

$$\begin{aligned} \int_{f^{-1}(A)} |\operatorname{adj} Df(x)|\, dx &= \int_{f^{-1}(A)\cap\Omega'} g(f(x))\, J_f(x)\, dx \\ &= \int_A g(y)\, dy. \end{aligned} \tag{3.4}$$

For $A = B$ this gives (3.3) and for other sets A it also implies $g \in L^1_{\mathrm{loc}}$. Hence $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ and from Theorem 3.2 we obtain that f^{-1} has finite outer distortion.

We will use Lemma 3.3 to prove (1.1). First let us notice that the Lusin (N) condition is valid on $f^{-1}(A)$ and therefore we can use (2.3) there. Indeed, let $S \subset f^{-1}(A)$ be a set of measure zero and let us find a Borel measurable set $S_1 \supset S$ of measure zero. We can use (2.2) for f^{-1} and $\eta = \chi_{S_1}$ and we obtain

$$\int_{f(S_1)} |J_{f^{-1}}| \leq |S_1| = 0.$$

Since $J_{f^{-1}} > 0$ on A it follows that $|f(S_1)| = 0$. Since f^{-1} is a mapping of finite distortion and each $W^{1,1}$ function is approximatively differentiable almost everywhere we obtain

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_E |Df^{-1}(y)|\, dy$$

and analogously

$$\int_{\tilde{E}} |\operatorname{adj} Df(x)|\, dx = \int_\Omega |\operatorname{adj} Df(x)|\, dx$$

since f is a mapping of finite inner distortion. Now we can use $|E \setminus A| = 0$, (2.3), (3.2), (2.1) and $|\tilde{E} \setminus f^{-1}(A)| = 0$ to obtain

$$\begin{aligned}\int_{f(\Omega)} |Df^{-1}(y)|\, dy &= \int_A |Df^{-1}(y)|\, dy \\ &= \int_{f^{-1}(A)} |Df^{-1}(f(x))|\, J_f(x)\, dx = \int_{f^{-1}(A)} |(Df(x))^{-1}| J_f(x)\, dx \\ &= \int_{f^{-1}(A)} |\operatorname{adj} Df(x)|\, dx = \int_\Omega |\operatorname{adj} Df(x)|\, dx.\end{aligned}$$

□

4. Construction of examples

In this section we use a notation $Q(c,r)$ for an open cube in $\mathbb{R}^{n-1}$ centered at c with edge length $2r$.

One of the main ingredients of the proof of Lemma 3.1 is the fact that homeomorphism $f \in W^{1,n-1}$ must satisfy the $(n-1)$-dimensional Lusin (N) condition on almost all hyperplanes. First we construct an auxiliary mapping that fails the Lusin (N) condition in $\mathbb{R}^{n-1}$. For a construction of a homeomorphism that does not satisfy the Lusin condition (N) we use Cantor type construction from [12] (see also [15],[5]).

Example 4.1. *Let $0 < \varepsilon < 1$ and $n \geq 3$. There is a homeomorphism $g \in W^{1,n-1-\varepsilon}((-1,1)^{n-1},(-1,1)^{n-1})$ such that $J_g \in L^\infty((-1,1)^{n-1})$ and $|\operatorname{adj} Dg| \in L^1((-1,1)^{n-1})$, but g does not satisfy the Lusin condition (N).*

Proof. By $\mathbb{V}$ we denote the set of 2^n vertices of the cube $[-1,1]^{n-1}$. The sets $\mathbb{V}^k = \mathbb{V} \times \ldots \times \mathbb{V}$, $k \in \mathbb{N}$, will serve as the sets of indices for our construction.

Let us denote

$$a_k = \frac{1}{k} \text{ and } b_k = \frac{1}{2}\Big(1 + \frac{1}{k^{n-1}}\Big). \tag{4.1}$$

Set $z_0 = \tilde{z}_0 = 0$ and let us define

$$r_k = a_k 2^{-k} \text{ and } \tilde{r}_k = b_k 2^{-k}. \tag{4.2}$$

It follows that $(-1,1)^{n-1} = Q(z_0, r_0)$ and further we proceed by induction. For $\boldsymbol{v} = [v_1, \ldots, v_k] \in \mathbb{V}^k$ we denote $\boldsymbol{w} = [v_1, \ldots, v_{k-1}]$ and we define

$$z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + \frac{1}{2} r_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^{k} r_{j-1} v_j,$$

$$Q'_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, \tfrac{r_{k-1}}{2}) \text{ and } Q_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, r_k).$$

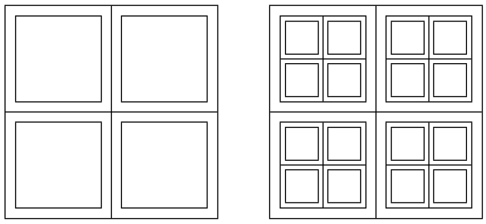

Fig. 1. Cubes $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^1$ and $\boldsymbol{v} \in \mathbb{V}^2$.

The number of the cubes $\{Q_{\boldsymbol{v}} : \ \boldsymbol{v} \in \mathbb{V}^k\}$ is $2^{(n-1)k}$. It is not difficult to find out that the resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}} =: C_A = C_a \times \ldots \times C_a$$

is a product of $n-1$ Cantor sets in $\mathbb{R}$. Moreover $\mathcal{L}_{n-1}(C_A) = 0$ since

$$\mathcal{L}_{n-1}\big(\bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}}\big) = 2^{(n-1)k}(2a_k 2^{-k})^{n-1} \overset{k\to\infty}{\to} 0.$$

Analogously we define

$$\tilde{z}_{\boldsymbol{v}} = \tilde{z}_{\boldsymbol{w}} + \frac{1}{2}\tilde{r}_{k-1}v_k = \tilde{z}_0 + \frac{1}{2}\sum_{j=1}^{k} \tilde{r}_{j-1}v_j,$$

$$\tilde{Q}'_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, \tfrac{\tilde{r}_{k-1}}{2}) \text{ and } \tilde{Q}_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, \tilde{r}_k).$$

The resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} \tilde{Q}_{\boldsymbol{v}} =: C_B = C_b \times \ldots \times C_b$$

satisfies $\mathcal{L}_{n-1}(C_B) > 0$ since $\lim_{k\to\infty} b_k > 0$. It remains to find a homeomorphism g which maps C_A onto C_B and satisfies our assumptions. Since $\mathcal{L}_{n-1}(C_A) = 0$ and $\mathcal{L}_{n-1}(C_B) > 0$ we will obtain that g does not satisfy the (N) condition.

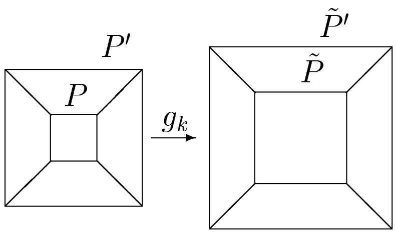

Fig. 2. The transformation of $Q' \setminus Q^{\circ}$ onto $\tilde{Q}' \setminus \tilde{Q}^{\circ}$

Again we will proceed by induction and we will find a sequence of homeomorphisms $g_k : (-1,1)^{n-1} \to (-1,1)^{n-1}$. We set $g_0(x) = x$ and for $k \in \mathbb{N}$ we define

$$g_k(x) = \begin{cases} g_{k-1}(x) & \text{for } x \notin \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q'_{\boldsymbol{v}} \\ g_{k-1}(z_{\boldsymbol{v}}) + (\alpha_k \|x - z_{\boldsymbol{v}}\| + \beta_k) \frac{x - z_{\boldsymbol{v}}}{\|x - z_{\boldsymbol{v}}\|} & \text{for } x \in Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^k \\ g_{k-1}(z_{\boldsymbol{v}}) + \frac{\tilde{r}_k}{r_k}(x - z_{\boldsymbol{v}}) & \text{for } x \in Q_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^k \end{cases}$$

where the constants α_k and β_k are given by

$$\alpha_k r_k + \beta_k = \tilde{r}_k \text{ and } \alpha_k \tfrac{r_{k-1}}{2} + \beta_k = \tfrac{\tilde{r}_{k-1}}{2}. \tag{4.3}$$

It is not difficult to find out that each g_k is a homeomorphism and maps

$$\bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}} \text{ onto } \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} \tilde{Q}_{\boldsymbol{v}}.$$

The limit $g(x) = \lim_{k\to\infty} g_k(x)$ is clearly one to one and continuous and therefore a homeomorphism. Moreover it is easy to see that g is differentiable almost everywhere, absolutely continuous on almost all lines parallel to coordinate axes and maps C_A onto C_B.

Let $k \in \mathbb{N}$ and $\boldsymbol{v} \in \mathbb{V}^k$. We need to estimate $Dg(x)$, $|\operatorname{adj} Dg|$ and $J_g(x)$ in the interior of the annulus $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$. Since

$$g(x) = g(z_{\boldsymbol{v}}) + (\alpha_k \|x - z_{\boldsymbol{v}}\| + \beta_k) \frac{x - z_{\boldsymbol{v}}}{\|x - z_{\boldsymbol{v}}\|}$$

there, we can use Lemma 2.1, $r_k \sim r_{k-1}$, $\tilde{r}_k \sim \tilde{r}_{k-1}$ (4.3), (4.2) and (4.1) to obtain

$$Dg(x) \sim \max\Big\{\frac{\tilde{r}_k}{r_k}, \alpha_k\Big\} \sim \max\Big\{k, \frac{1}{k^{n-2}}\Big\} \sim k,$$

$$|\operatorname{adj} Dg(x)| \sim |Df(x)| \Big(\frac{\tilde{r}_k}{r_k}\Big)^{n-3} \sim k^{n-2} \text{ and } J_g(x) \sim \alpha_k \Big(\frac{\tilde{r}_k}{r_k}\Big)^{n-2} \sim 1.$$

It follows that $J_g \in L^\infty((-1,1)^{n-1})$. Moreover we can estimate

$$\mathcal{L}_{n-1}(Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}) = (r_{k-1})^{n-1} - (2r_k)^{n-1} \sim 2^{-k(n-1)} \frac{1}{k^n}$$

and we have $2^{(k-1)n}$ annuli like that. Therefore

$$\begin{aligned} \int_{Q_0} |Dg(x)|^{n-1-\varepsilon}\, dx &\leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{v} \in \mathbb{V}^k} \int_{Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}} |Dg(x)|^{n-1-\varepsilon}\, dx \\ &\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} k^{n-1-\varepsilon} < \infty \end{aligned}$$

$$\begin{aligned} \text{and } \int_{Q_0} |\operatorname{adj} Dg(x)|\, dx &\leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{v} \in \mathbb{V}^k} \int_{Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}} |\operatorname{adj} Dg(x)|\, dx \\ &\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} k^{n-2} < \infty. \end{aligned}$$

□

Proof of Example 1.2. In this example we will use notation and results from Example 4.1. Set

$$f(x) = \Big[g_1([x_1,\dots,x_{n-1}]),\dots,g_{n-1}([x_1,\dots,x_{n-1}]),e^{-x_n}\Big].$$

Furher we define

$$\Omega = \big(C_A\times(0,\infty)\big)\cup\bigcup_{k=1}^{\infty}\Big(\bigcup_{\boldsymbol{v}\in\mathbb{V}^k} Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}\Big)\times(0,\log(k+1)).$$

Clearly f is a homeomorphism and both f and f^{-1} are differentiable almost everywhere. Moreover it is easy to check, that $\Omega\subset(-1,1)^{n-1}\times(0,\infty)$ is an open set.

The matrix Df has a special form, because only one term in the last column and in the last row is non-zero. This is the term $\frac{\partial f_n}{\partial x_n}$ and therefore it is easy to check that

$$|\operatorname{adj} Df(x)|\sim\max\Big\{|J_g(\tilde{x})|,|\operatorname{adj} Dg(\tilde{x})|\Big|\frac{\partial e^{-x_n}}{\partial x_n}\Big|\Big\},$$

where $\tilde{x}=[x_1,\dots,x_{n-1}]$. From

$$\begin{aligned}\mathcal{L}_n(\Omega)&=\sum_{k\in\mathbb{N}}\sum_{\boldsymbol{v}\in\mathbb{V}^k}\mathcal{L}_{n-1}\big(Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}\big)\log(k+1)\\&=\sum_{k\in\mathbb{N}}2^{(k-1)n}2^{-k(n-1)}\frac{1}{k^n}\log(k+1)<\infty\end{aligned}$$

and $|J_g|\in L^\infty((-1,1)^{n-1})$ we obtain $|J_g(\tilde{x})|\in L^1(\Omega)$. Further

$$\int_\Omega|\operatorname{adj} Dg(\tilde{x})|\Big|\frac{\partial e^{-x_n}}{\partial x_n}\Big|\,dx\le\int_{(-1,1)^{n-1}}|\operatorname{adj} Dg|\,\int_0^\infty e^{-x_n}\,dx_n<\infty$$

and hence $|\operatorname{adj} Df|\in L^1(\Omega)$. Moreover

$$Df(x)=\max\{|Dg(\tilde{x})|,|\tfrac{\partial e^{-x_n}}{\partial x_n}|\}\sim|Dg(\tilde{x})|$$

and therefore

$$\begin{aligned}\int_\Omega|Df(x)|^{n-1-\varepsilon}\,dx&\le\sum_{k=1}^{\infty}\sum_{\boldsymbol{v}\in\mathbb{V}^k}\Big(\int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}}|Dg(\tilde{x})|^{n-1-\varepsilon}\,d\tilde{x}\Big)\,\log(k+1)\\&\le C\sum_{k=1}^{\infty}2^{(k-1)n}2^{-k(n-1)}\frac{1}{k^n}k^{n-1-\varepsilon}\log(k+1)<\infty.\end{aligned}$$

Since $C_A\times(0,\infty)\subset\Omega$ we obtain that

$$f^{-1}\big(\{[y,t]\in f(\Omega):\ t\in(0,1)\}\big)=g^{-1}(y)\times(0,\infty)\text{ for every }y\in C_B$$

and thus

$$\int_0^1|\nabla f^{-1}(y,t)|\,dt\ge\int_0^1\Big|\frac{\partial f^{-1}}{\partial t}(y,t)\Big|\,dt=\infty.$$

Since $\mathcal{L}_{n-1}(C_B)>0$ we obtain that $|\nabla f^{-1}|\notin L^1(f(\Omega))$. $\square$

Remark 4.2. *Let us note that the fact that Ω is unbounded is not essential for our arguments, it only makes them simpler. It would be possible to twist our Ω and to obtain a bounded domain with the same properties.*

5. Sharpness on the Orlicz scale

Lemma 5.1. *Let $h : (0,1) \to (0,\infty)$ be an increasing function such that $\lim_{t\to 0+} h(t) = 0$. Then there is a function $f : (0,1) \to (0,\infty)$ such that $\lim_{t\to 0+} f(t) = 0$,*

$$\int_0^1 \frac{f(t)}{t}\,dt = \infty \text{ and } \int_0^1 \frac{f(t)h(t)}{t}\,dt < \infty.$$

Proof. We can easily find an increasing differentiable function $h_1 \geq h$ that satisfies $\lim_{t\to 0+} h_1(t) = 0$ and $\lim_{t\to 0+} \frac{th_1'(t)}{h_1(t)} = 0$ which is some sort of strong concavity near 0. Thus we may assume without loss of generality that h is differentiable and that the function

$$f(t) := \frac{th'(t)}{h(t)} \text{ satisfies } \lim_{t\to 0+} f(t) = 0. \tag{5.1}$$

An elementary computation gives us

$$\int_0^1 \frac{f(t)}{t}\,dt = \int_0^1 \frac{h'(t)}{h(t)}\,dt = \Big[\log h(t)\Big]_{t=0}^{t=1} = \infty$$

$$\text{and} \int_0^1 \frac{f(t)h(t)}{t}\,dt = \int_0^1 h'(t)\,dt = \Big[h(t)\Big]_{t=0}^{t=1} < \infty.$$

□

Proof of Example 1.3. We write $\mathbf{e}_i$ for the i-th unit vector in $\mathbb{R}^n$, i.e. the vector with 1 on the i-th place and 0 everywhere else. Given $x = [x_1, \dots, x_n] \in \mathbb{R}^n$ we denote $\tilde{x} = [x_1, \dots, x_{n-1}] \in \mathbb{R}^{n-1}$ and $\|\tilde{x}\| = \sqrt{x_1^2 + \ldots + x_{n-1}^2}$.

From Lemma 5.1 we can find a function $a : (0,\infty) \to (0,\infty)$ such that

$$\lim_{t\to 0+} a(t) = 0,$$

$$\int_0^1 \frac{a^{n-1}(t)}{t} dt = \infty \quad \text{and} \tag{5.2}$$

$$\int_0^1 \frac{a^{n-1}(t)}{t} g\Big(\frac{1}{\sqrt{t}}\Big) dt < \infty. \tag{5.3}$$

Without loss of generality we may also suppose that

$$\frac{1}{\log^{\frac{2}{n-1}} \frac{1}{t}} \leq a(t) \text{ for every } t \in (0, \tfrac{1}{2}) \tag{5.4}$$

since the integral in (5.2) is finite for the left hand side. Therefore it is easy to see that without loss of generality we can also assume that a is increasing and concave.

Set

$$f(x)=\sum_{i=1}^{n-1}\mathbf{e}_i\frac{x_i}{\|\tilde{x}\|}a(\|\tilde{x}\|)+\mathbf{e}_n\Big(x_n+\|\tilde{x}\|\sin\big(\tfrac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\big)\Big)$$

if $\|\tilde{x}\|>0$ and $f(x)=\mathbf{e}_n x_n$ if $\|\tilde{x}\|=0$. Our mapping f is clearly continuous and it is easy to check that f is a one-to-one map since

$$\begin{aligned}\frac{x_i}{\|\tilde{x}\|}a(\|\tilde{x}\|)=\frac{z_i}{\|\tilde{z}\|}a(\|\tilde{z}\|)\text{ for every }i\in\{1,\dots,n-1\}\Rightarrow\\ \Rightarrow a(\|\tilde{x}\|)=a(\|\tilde{z}\|)\text{ which implies }\|\tilde{x}\|=\|\tilde{z}\|\text{ and hence}\\ x_i=z_i\text{ for every }i\in\{1,\dots,n-1\}.\end{aligned}$$

Therefore f is a homeomorphism.

By Lemma 2.1 we obtain that the partial derivatives of f_i, $i\in\{1,\dots,n-1\}$, are smaller than

$$C\max\Big\{\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|},a'(\|\tilde{x}\|)\Big\}\sim C\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|},\tag{5.5}$$

since a is concave and $a(0)=0$. Moreover,

$$\begin{aligned}\frac{\partial f_n(x)}{\partial x_1}=&x_1\|\tilde{x}\|^{-1}\sin\big(\tfrac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\big)\\ &+\|\tilde{x}\|\Big(\frac{a'(\|\tilde{x}\|)x_1}{\|\tilde{x}\|^2}-\frac{a(\|\tilde{x}\|)x_1}{\|\tilde{x}\|^3}\Big)\cos\big(\tfrac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\big)\end{aligned}\tag{5.6}$$

can be also bounded by (5.5). Analogously we can bound other derivatives of f_n and therefore we can use substitution to spherical coordinates in $\mathbb{R}^{n-1}$ to obtain

$$\begin{aligned}\int_{B(0,1)}|Df(x)|^{n-1}g(|Df(x)|)\,dx&\leq C\int_{B(0,1)}\frac{a(\|\tilde{x}\|)^{n-1}}{\|\tilde{x}\|^{n-1}}g\Big(C\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\Big)\,dx\\ &\leq C\int_0^1\frac{a(t)^{n-1}}{t^{n-1}}g\Big(C\frac{a(t)}{t}\Big)t^{n-2}\,dt.\end{aligned}$$

From (5.4) we can find $\varepsilon>0$ such that for every $t\in(0,\varepsilon)$ we have $C\frac{a(t)}{t}\geq\frac{1}{\sqrt{t}}$ and therefore the last integral is finite by (5.3) and the condition (1.2) follows.

The inverse of f is given by

$$f^{-1}(y)=\sum_{i=1}^{n-1}\mathbf{e}_i\frac{y_i}{\|\tilde{y}\|}a^{-1}(\|\tilde{y}\|)+\mathbf{e}_n\Big(y_n-a^{-1}(\|\tilde{y}\|)\sin\big(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\big)\Big)$$

if $\|\tilde{y}\|>0$ and $f^{-1}(y)=\mathbf{e}_n y_n$ if $\|\tilde{y}\|=0$. The differential of f^{-1} is clearly continuous outside the segment $\{[0,\cdots,0,t]:t\in\mathbb{R}\}$.

Analogously to (5.6) we obtain

$$\begin{aligned}\frac{\partial(f^{-1})_n(y)}{\partial y_1} =&(a^{-1})'(\|\tilde{y}\|)y_1\|\tilde{y}\|^{-1}\sin\bigl(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\bigr)\\ &+a^{-1}(\|\tilde{y}\|)\Bigl(\frac{y_1}{\|\tilde{y}\|a^{-1}(\|\tilde{y}\|)}-\frac{y_1(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)^2}\Bigr)\cos\bigl(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\bigr).\end{aligned}$$

It follows that we can find $\delta>0$ such that

$$\Bigl|\frac{\partial(f^{-1})_n(y)}{\partial y_1}\Bigr|\geq C\|\tilde{y}\|\frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)} \tag{5.7}$$

for every

$$y\in S:=\bigl\{y\in B(0,\delta):\ y_1>\tfrac{1}{2}\|\tilde{y}\|,\ |\cos\bigl(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\bigr)|\geq\tfrac{\sqrt{2}}{2}\bigr\}.$$

Here we have also used the fact that (5.4) gives us

$$a^{-1}(y)\leq\exp\bigl(-\tfrac{1}{y^{\frac{n-1}{2}}}\bigr)\text{ for small enough } y.$$

Clearly $\mathcal{L}_n(S)=C\mathcal{L}_n(G)$ for

$$G:=\bigl\{y\in B(0,\delta):\ \bigl|\cos\bigl(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\bigr)\bigr|\geq\tfrac{\sqrt{2}}{2}\bigr\}$$

and thus we can use (5.7) to obtain

$$\begin{aligned}\int_{f(B(0,1))}|Df^{-1}(y)|\,dy&\geq\int_S\Bigl|\frac{\partial(f^{-1})_n(y)}{\partial y_1}\Bigr|\,dy\\ &\geq C\int_G\|\tilde{y}\|\frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\,dy.\end{aligned} \tag{5.8}$$

Now let us consider a mappings

$$h(x)=\sum_{i=1}^{n-1}\mathbf{e}_i\frac{x_i}{\|\tilde{x}\|}a(\|\tilde{x}\|)+\mathbf{e}_n\Bigl(x_n+\|\tilde{x}\|\cos\bigl(\tfrac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\bigr)\Bigr)$$

if $\|\tilde{x}\|>0$ and $h(x)=\mathbf{e}_nx_n$ if $\|\tilde{x}\|=0$. Analougously as above we obtain that h is a homeomorphism that satisfies (1.2) and that

$$\int_{h(B(0,1))}|Dh^{-1}(y)|\,dy\geq C\int_{\tilde{G}}\|\tilde{y}\|\frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\,dy \tag{5.9}$$

where

$$\tilde{G}=\bigl\{y\in B(0,\delta):\ \bigl|\sin\bigl(\tfrac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\bigr)\bigr|\geq\tfrac{\sqrt{2}}{2}\bigr\}$$

for some possibly smaller δ. By the formula of change of variables and (5.2) we obtain that

$$\begin{aligned}\int_{B(0,\delta)}\|\tilde{y}\|\frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\,dy&\geq C\int_0^\delta s\frac{(a^{-1})'(s)}{a^{-1}(s)}s^{n-2}\,ds\\ &\geq C\int_0^{a^{-1}(\delta)}a(t)\frac{1}{t}a(t)^{n-2}\,dt=\infty.\end{aligned} \tag{5.10}$$

From (5.8), (5.9), $G \cup \tilde{G} = B(0,\delta)$ and (5.10) we obtain that either $\nabla f \notin L^1$ or $\nabla h \notin L^1$ which is the desired conclusion. □

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