Symmetries in differential geometry

Petr Zima

Charles University,
Prague, Czech Republic

Jarní škola Katedry algebry,
April 6–9, 2017

Basic model

Left and right action

Let G be a group. We have the canonical left and right action on G,

$$L_g(g') = g\,g', \qquad R_g(g') = g'\,g, \qquad \forall g, g' \in \mathrm{G}.$$

- The left and right actions **commute**, $L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}$.
- These two actions are **balanced**, i.e. the left action exhausts the full set of commuting maps for the right action and vice versa.

Basic model

Left and right action

Let G be a group. We have the canonical left and right action on G,

$$L_g(g') = g\,g', \qquad R_g(g') = g'\,g, \qquad \forall g, g' \in \mathrm{G}.$$

- The left and right actions **commute**, $L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}$.
- These two actions are **balanced**, i.e. the left action exhausts the full set of commuting maps for the right action and vice versa.

Coset space

Further let $\mathrm{H} \leq \mathrm{G}$ be a subgroup and consider the left coset space $\mathsf{M} = \mathrm{G}/\mathrm{H}$ together with the canonical projection $p\colon \mathrm{G} \to \mathsf{M}$.

- The left action L_g projects down yielding $\underline{L}_g\colon \mathsf{M} \to \mathsf{M}$.
- The right action R_g does not project down to M.
- H acts simply transitively on the cosets $g\mathrm{H}$ from the right.

If we consider ($\mathrm{G} \to \mathsf{M}$, R_g for $g \in G$) to be the “given structure” on M then the left action $\underline{L}_g$ gives all its **symmetries** (≡ automorphisms).

Smooth manifolds

Let M be a smooth manifold. We have:

- The associative and commutative algebra $C(M)$ of smooth ($\equiv$ infinitely differentiable) real functions on M.
- The $C(M)$-module $\mathcal{X}(M)$ of **vector fields** $\equiv$ derivations in $C(M)$ $\equiv$ $\mathbb{R}$-linear maps $X\colon C(M) \to C(M)$ satisfying the **Leibniz rule**,

$$X(f\,f') = X(f)\,f' + f\,X(f'), \qquad \forall f, f' \in C(M).$$

Smooth manifolds

Let M be a smooth manifold. We have:

- The associative and commutative algebra $C(M)$ of smooth ($\equiv$ infinitely differentiable) real functions on M.
- The $C(M)$-module $\mathcal{X}(M)$ of **vector fields** $\equiv$ derivations in $C(M)$ $\equiv$ $\mathbb{R}$-linear maps $X\colon C(M) \to C(M)$ satisfying the **Leibniz rule**,

$$X(f\,f') = X(f)\,f' + f\,X(f'), \qquad \forall f, f' \in C(M).$$

- At each point $x \in M$ the vector space $\mathcal{T}_x(M)$ of **tangent vectors** $\equiv$ $\mathbb{R}$-linear maps $v\colon C(M) \to \mathbb{R}$ satisfying the Leibniz rule at x,

$$v(f\,f') = v(f)\,f'(x) + f(x)\,v(f'), \qquad \forall f, f' \in C(M).$$

  We identify the "geometric" notion of tangent vector with the corresponding directional derivative.
- Evaluating a vector field X at x yields a tangent vector X_x,

$$X_x(f) = (X(f))(x), \qquad \forall f \in C(M).$$

Lie groups and infinitesimal actions

Tangent map

From now on we assume all manifolds and maps between them smooth. The **tangent map** (≡ differential) of a map $F\colon \mathsf{M} \longrightarrow \mathsf{N}$ is a collection of linear maps between the tangent spaces $\mathrm{d}F_x\colon \mathcal{T}_x(\mathsf{M}) \longrightarrow \mathcal{T}_{F(x)}(\mathsf{N})$,

$$(\mathrm{d}F_x(v))(f) = v(f \circ F), \qquad \forall x \in M, v \in \mathcal{T}_x(\mathsf{M}),\ f \in \mathcal{C}(\mathsf{N}).$$

Lie groups and infinitesimal actions

Tangent map

From now on we assume all manifolds and maps between them smooth. The **tangent map** ($\equiv$ differential) of a map $F\colon \mathsf{M} \longrightarrow \mathsf{N}$ is a collection of linear maps between the tangent spaces $\mathrm{d}F_x\colon \mathcal{T}_x(\mathsf{M}) \longrightarrow \mathcal{T}_{F(x)}(\mathsf{N})$,

$$(\mathrm{d}F_x(v))(f) = v(f \circ F), \qquad \forall x \in M, v \in \mathcal{T}_x(\mathsf{M}),\ f \in \mathcal{C}(\mathsf{N}).$$

Lie group

$\equiv$ a smooth manifold G with a smooth group structure. Let us denote

- $\mathfrak{g} = \mathcal{T}_e(\mathrm{G})$, the tangent space at the identity element $e \in \mathrm{G}$,
- and further consider a left action $\lambda\colon \mathrm{G} \times \mathsf{M} \longrightarrow \mathsf{M}$.

Now we differentiate the partially evaluated map $\lambda(\cdot, x)\colon \mathrm{G} \longrightarrow \mathsf{M}$ and define the **infinitesimal action** as a linear map $\lambda'\colon \mathfrak{g} \longrightarrow \mathcal{X}(\mathsf{M})$,

$$\lambda'(V)_x = \mathrm{d}\lambda(\cdot, x)_e(V), \qquad \forall V \in \mathfrak{g}, x \in \mathsf{M}.$$

Similarly we define $\rho'\colon \mathfrak{g} \longrightarrow \mathcal{X}(\mathsf{M})$ for a right action $\rho\colon \mathsf{M} \times \mathrm{G} \longrightarrow \mathsf{M}$.

Lie algebras

≡ A vector space $\mathfrak{a}$ with a bilinear **bracket** $[,]\colon \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ which is

1. alternating and hence anti-commutative,

$$[a, a] = 0, \qquad \Longrightarrow \qquad [a, b] = -[b, a], \qquad \forall a, b \in \mathfrak{a};$$

2. and satisfies the **Jacobi identity**,

$$[a, [b, c]] + [c, [a, b]] + [b, [a, c]] = 0, \qquad \text{or equivalently,}$$
$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \qquad \forall a, b, c \in \mathfrak{a}.$$

Note how the second form of the identity resembles the Leibniz rule.

Lie algebras

≡ A vector space $\mathfrak{a}$ with a bilinear **bracket** $[\,,\,]\colon \mathfrak{a}\times\mathfrak{a}\longrightarrow\mathfrak{a}$ which is

1. alternating and hence anti-commutative,

$$[a,a]=0, \qquad \Longrightarrow \qquad [a,b]=-[b,a], \qquad \forall a,b\in\mathfrak{a};$$

2. and satisfies the **Jacobi identity**,

$$[a,[b,c]]+[c,[a,b]]+[b,[a,c]]=0, \qquad \text{or equivalently,}$$
$$[a,[b,c]]=[[a,b],c]+[b,[a,c]], \qquad \forall a,b,c\in\mathfrak{a}.$$

Note how the second form of the identity resembles the Leibniz rule.

Examples

- An associative algebra A equipped with the **commutator**,

$$[a,b]=a\,b-b\,a, \qquad \forall a,b\in\mathrm{A}.$$

- The space Der(A) of **derivations** in any algebra A, in particular, the vector fields $\mathcal{X}(M)$ on a smooth manifold M.

Invariant vector fields

Let G be a Lie group and recall the left and right actions of G on itself. We consider the infinitesimal actions $L', R' : \mathfrak{g} \to \mathcal{X}(\mathrm{G})$ which assign vector fields to each $V \in \mathfrak{g}$.

- The left and right actions commute hence $R'(V)$ is preserved by L_g.

$\Rightarrow$ We call the vector fields $R'(V)$ **left-invariant** and vice versa.

$\Rightarrow$ The left and right invariant fields themselves commute.

Invariant vector fields

Let G be a Lie group and recall the left and right actions of G on itself. We consider the infinitesimal actions $L', R' : \mathfrak{g} \to \mathcal{X}(\mathrm{G})$ which assign vector fields to each $V \in \mathfrak{g}$.

- The left and right actions commute hence $R'(V)$ is preserved by L_g.

⇒ We call the vector fields $R'(V)$ **left-invariant** and vice versa.

⇒ The left and right invariant fields themselves commute.

- Moreover, $R'(\mathfrak{g})$ are *all* the vector fields invariant with respect to L_g and the invariance is preserved by the bracket in $\mathcal{X}(\mathrm{G})$.

⇒ $\mathfrak{g}$ becomes a Lie algebra by pulling back the bracket from $\mathcal{X}(\mathrm{G})$,

$$[U, V]_{\mathfrak{g}} = R'^{-1}([R'(U), R'(V)]_{\mathcal{X}}), \qquad \forall U, V \in \mathfrak{g}.$$

called the **Lie algebra of G**.

Invariant vector fields

Let G be a Lie group and recall the left and right actions of G on itself. We consider the infinitesimal actions $L', R' : \mathfrak{g} \to \mathcal{X}(\mathrm{G})$ which assign vector fields to each $V \in \mathfrak{g}$.

- ▸ The left and right actions commute hence $R'(V)$ is preserved by L_g.
- ⇒ We call the vector fields $R'(V)$ **left-invariant** and vice versa.
- ⇒ The left and right invariant fields themselves commute.
- ▸ Moreover, $R'(\mathfrak{g})$ are *all* the vector fields invariant with respect to L_g and the invariance is preserved by the bracket in $\mathcal{X}(\mathrm{G})$.
- ⇒ $\mathfrak{g}$ becomes a Lie algebra by pulling back the bracket from $\mathcal{X}(\mathrm{G})$,

$$[U, V]_{\mathfrak{g}} = R'^{-1}([R'(U), R'(V)]_{\mathcal{X}}), \qquad \forall U, V \in \mathfrak{g}.$$

  called the **Lie algebra of G**.
- ▸ Any infinitisemal right action $\rho' : \mathfrak{g} \to \mathcal{X}(\mathrm{M})$ preserves the bracket $\equiv \rho'$ is a **Lie algebra homomorphism**.
- ⇒ In particular, the conjugation in G induces the so called **adjoint representation** $\mathrm{Ad}\colon \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ of G on $\mathfrak{g}$.

Principal bundles

Let M be a manifold and H be a Lie group. An **H-principal bundle** over M is a manifold P together with

1. a projection $p\colon \mathsf{P} \to \mathsf{M}$ such that $\mathrm{d}p_u$ is surjective for all $u \in \mathsf{P}$;
2. a *simple* right action R_h of H on P such that the orbits of R_h are exactly the **fibers** $\mathsf{P}_x = p^{-1}(x)$, $x \in \mathsf{M}$.

Note that $\mathsf{P} \to \mathsf{M}$ is a generalization of the left coset space $G \to \mathsf{G}/\mathsf{H}$.

Principal bundles

Let M be a manifold and H be a Lie group. An **H-principal bundle** over M is a manifold P together with

1. a projection $p\colon \mathsf{P} \to \mathsf{M}$ such that $\mathrm{d}p_u$ is surjective for all $u \in \mathsf{P}$;
2. a *simple* right action R_h of H on P such that the orbits of R_h are exactly the **fibers** $\mathsf{P}_x = p^{-1}(x)$, $x \in \mathsf{M}$.

Note that $\mathsf{P} \to \mathsf{M}$ is a generalization of the left coset space $G \to \mathsf{G}/\mathsf{H}$.

Tensorial quantities

Let $\mathsf{P} \to \mathsf{M}$ be an H-principal bundle and $\lambda\colon \mathsf{H} \to \mathrm{Aut}_{\mathbb{R}}(\mathsf{V})$ a representation of H on a vector space V. We consider H-invariant V-valued functions on P,

$$C(\mathsf{P}, \mathsf{V})^{\mathsf{H}} = \{f\colon \mathsf{P} \to \mathsf{V} \mid f \circ R_h^{-1} = \lambda(h) \circ f,\ \forall h \in \mathsf{H}\}.$$

Such functions correspond to "tensorial quantities of type λ" on M.

- An element $u \in \mathsf{P}_x$ can be understood as a **frame of reference** $\equiv$ an infinitesimal coordinate system at $x \in \mathsf{M}$.
- This corresponds to the classical description of **tensors** as quantities which transform in an appropriate way under coordinate change.

Cartan geometries

Let H be a Lie group, $\mathfrak{h}$ its Lie algebra and $\mathfrak{g}$ another Lie algebra such that $\mathfrak{h} \leq \mathfrak{g}$ is its subalgebra. A **Cartan geometry** of type $(\mathfrak{g}, \mathsf{H})$ on M is a triple $(\mathsf{M}, \mathsf{P}, r)$ where

1. P is a principal H-bundle over M with right action R_h;
2. $r \colon \mathfrak{g} \to \mathcal{X}(\mathsf{P})$ is a $\mathbb{R}$-linear map such that
3. the corestriction on $\mathcal{T}_u(\mathsf{P})$ is a linear isomorphism for each $u \in \mathsf{P}$;
4. it is transformed by R_h according to the adjoint representation

$$\mathrm{d}R_h(r(V)_u) = r(\mathrm{Ad}(h)^{-1}\, V)_{R_h(u)}, \qquad \forall V \in \mathfrak{g}, h \in \mathsf{H}, u \in \mathsf{P};$$

5. and its restriction on $\mathfrak{h}$ is just the infinitesimal action R'.

The **homogeneous model** is the quotient $\mathrm{G} \to \mathrm{G}/\mathsf{H}$ with the infinitesimal right action of $\mathfrak{g}$ where G is a suitable Lie group with Lie algebra $\mathfrak{g}$.

Cartan geometries

Let H be a Lie group, $\mathfrak{h}$ its Lie algebra and $\mathfrak{g}$ another Lie algebra such that $\mathfrak{h} \leq \mathfrak{g}$ is its subalgebra. A **Cartan geometry** of type $(\mathfrak{g}, \mathsf{H})$ on M is a triple $(\mathsf{M}, \mathsf{P}, r)$ where

1. P is a principal H-bundle over M with right action R_h;
2. $r : \mathfrak{g} \to \mathcal{X}(\mathsf{P})$ is a $\mathbb{R}$-linear map such that
3. the corestriction on $\mathcal{T}_u(\mathsf{P})$ is a linear isomorphism for each $u \in \mathsf{P}$;
4. it is transformed by R_h according to the adjoint representation

$$\mathrm{d}R_h(r(V)_u) = r(\mathrm{Ad}(h)^{-1}\, V)_{R_h(u)}, \qquad \forall V \in \mathfrak{g}, h \in \mathsf{H}, u \in \mathsf{P};$$

5. and its restriction on $\mathfrak{h}$ is just the infinitesimal action R'.

The **homogeneous model** is the quotient $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ with the infinitesimal right action of $\mathfrak{g}$ where G is a suitable Lie group with Lie algebra $\mathfrak{g}$.

- This notion covers a wide class of sufficiently rigid geometric structures, e.g. Riemannian, conformal or projective geometry.
- The vector fields $r(V)$ are not projectable down to M.
- On the homogeneous model r is a Lie algebra homomorphism.

Tractors

The condition that r is an isomorphism on the fibers allows us to identify the vector fields $\xi \in \mathcal{X}(\mathsf{P})$ with functions $f \in \mathcal{C}(\mathsf{P}, \mathfrak{g})$,

$$\xi_u = r(f(u))_u, \qquad \forall u \in \mathsf{P}.$$

- We call the inverses $\omega_u = (r_u)^{-1} \colon \mathcal{T}_u(\mathrm{G}) \to \mathfrak{g}$ **Cartan connection**.
- The vector fields on P which are projectable down to M are exactly the subspace $\mathcal{A}(\mathsf{M}) = \mathcal{X}(\mathsf{P})^{\mathsf{H}} \leq \mathcal{X}(\mathsf{P})$ of R_h-invariant vector fields.
- These invariant vector fields are tensorial of type Ad, they are called **adjoint tractors**.

Tractors

The condition that r is an isomorphism on the fibers allows us to identify the vector fields $\xi \in \mathcal{X}(\mathsf{P})$ with functions $f \in C(\mathsf{P}, \mathfrak{g})$,

$$\xi_u = r(f(u))_u, \qquad \forall u \in \mathsf{P}.$$

- We call the inverses $\omega_u = (r_u)^{-1} \colon \mathcal{T}_u(\mathsf{G}) \to \mathfrak{g}$ **Cartan connection**.
- The vector fields on P which are projectable down to M are exactly the subspace $\mathcal{A}(\mathsf{M}) = \mathcal{X}(\mathsf{P})^{\mathsf{H}} \leq \mathcal{X}(\mathsf{P})$ of R_h-invariant vector fields.
- These invariant vector fields are tensorial of type Ad, they are called **adjoint tractors**.
- The tangent vector fields $\mathcal{X}(\mathsf{M})$ on the base $\mathcal{X}$ are tensorial of type $\underline{\mathrm{Ad}} \colon \mathsf{H} \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$.
- We have a projection $\Pi \colon \mathcal{A}(\mathsf{M}) \to \mathcal{X}(\mathsf{M})$ given by $\mathrm{d}p$. On the fibers it corresponds to the canonical projection $\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.
- In general, **tractors** are tensorial quantities of type $\lambda \colon \mathsf{H} \to \mathrm{Aut}_{\mathbb{R}}(V)$ such that $\lambda' \colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(\mathsf{V})$ can be extended to the whole $\mathfrak{g}$.