Circular units of abelian fields with four ramified primes

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Overview

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   - General definitions
   - The cases of one, two or three ramified primes
2. Finding a basis of D^+ for $|P| = 4$
   - Notation and assumptions
   - General strategy
   - A special subcase
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Circular units appear in many situations in algebraic number theory, because in some sense they are a good approximation of the full group of units of a given abelian field, which is very hard to describe explicitly. They are also closely related to the class group of the respective field.

The problem is that a $\mathbb{Z}$-basis of the group of circular units is known only in a few very special cases, for example when the abelian field is cyclotomic, has at most two ramified primes, or has three ramified primes and satisfies some other conditions.

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Let k be a real abelian field, P be the set of ramified primes of k, K be the genus field of k in the narrow sense and $G = \mathrm{Gal}(K/\mathbb{Q})$, so that $G \cong \prod_{p \in P} T_p$, where T_p is the inertia group for the prime p.

Then D^+, the non-torsion part of the group D of circular numbers of k (using Lettl's modification of Sinnott's definition), has one generator η_I for each nonempty subset $I \subseteq P$.

More explicitly, if we let K_i be the largest subfield of K in which p_i is the only ramified prime for any $i \in I$, we have

$$\eta_I = \mathrm{N}_{\mathbb{Q}(\zeta_{\mathrm{cond}(\prod_{i\in I} K_i)})/(\prod_{i\in I} K_i)\cap k}\left(1 - \zeta_{\mathrm{cond}(\prod_{i\in I} K_i)}\right).$$

It turns out that D^+ is a $\mathbb{Z}[G]$-module of $\mathbb{Z}$-rank $[k : \mathbb{Q}] + |P| - 1$.

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The generators of D^+ are subject to norm relations (which can be obtained by computing the norm of the generators to a subfield with less ramified primes) and for $|P| \geq 3$, also to the so-called Ennola relations, which are highly nontrivial relations that are not consequences of the norm relations.

Our goal will be to find a basis of D^+ (it can then be easily modified in order to obtain a basis of the group of circular units). In general, this is very difficult, so we will need to restrict ourselves to a special family of fields. It's clear from the definition of D^+ that the complexity of the problem depends on the number of ramified primes in k.

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If $|P| = 1$, the set of all conjugates of η already forms a basis of D^+, since the rank of D^+ is exactly $[k : \mathbb{Q}]$.

If $|P| = 2$, the situation is a little more complicated, but still quite easily managable. This is dealt with in the papers of K.Dohmae.

For $|P| = 3$, R.Kučera and A.Salami have recently constructed a basis under the assumption that the relative Galois group $\mathrm{Gal}(K/k)$ is cyclic, as well as the inertia groups for all ramified primes. However, this basis is too complicated to describe here, mainly due to the presence of Ennola relations.

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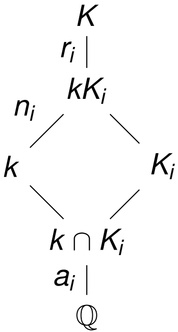
Now let $P = \{p_1, p_2, p_3, p_4\}$, $m := [K : k]$ and for $i \in \{1, 2, 3, 4\}$, let $a_i := [k \cap K_i : \mathbb{Q}]$, $r_i := [K : kK_i]$ and $n_i := \frac{m}{r_i}$.



We will assume that $H := \mathrm{Gal}(K/k) = \langle \tau \rangle$ is cyclic, as well as the inertia subgroups $T_i = \langle \sigma_i \rangle$ (without loss of generality, we have $\tau = \sigma_1^{a_1} \sigma_2^{a_2} \sigma_3^{a_3} \sigma_4^{a_4}$).

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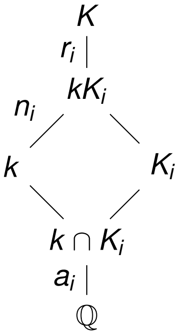

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Now let $R_i = \sum_{u=0}^{a_i-1} \sigma_i^u$ and $N_i = \sum_{u=0}^{m-1} \sigma_i^{ua_i}$. If we regard these as elements of $\mathbb{Z}[G/H]$, then $R_i N_i$ is the norm operator from k to the maximal subfield ramified at less primes.

In $\mathbb{Z}[G/H]$, we also have

$$N_4 = \sum_{u=0}^{m-1} \sigma_1^{ua_1} \sigma_2^{ua_2} \sigma_3^{ua_3}.$$

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To construct the basis of D^+, we can take the union of all bases for the fields

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(which have three ramified primes) and add in

$$a_1 a_2 a_3 a_4 \frac{m^3}{r_1 r_2 r_3 r_4} - \sum_{i,j,l} a_i a_j a_l \frac{m^2}{r_i r_j r_l} + \sum_{i,j} a_i a_j \gcd(r_i, r_j) \frac{m}{r_i r_j} - \sum_i a_i + 1$$

conjugates of the highest generator $\eta := \eta_{\{1,2,3,4\}}$.

To show that we will obtain a basis, we need to be able to generate all the missing conjugates of η. We will use the norm relations to do this.

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Now we will focus on the subcase $a_1 = a_2 = a_3 = r_4 = 1$. Here we have

$$\mathrm{Gal}(k/\mathbb{Q}) \cong G/H \cong \{\mathrm{res}_{K/k}\left(\sigma_1^{x_1}\sigma_2^{x_2}\sigma_3^{x_3}\sigma_4^{x_4}\right); 0 \le x_1 < n_1, \\ 0 \le x_2 < n_2, 0 \le x_3 < n_3, 0 \le x_4 < a_4\},$$

where each automorphism of k determines the quadruple (x_1, x_2, x_3, x_4) uniquely.

Since the conjugates of η correspond to the elements of $\mathrm{Gal}(k/\mathbb{Q})$, we can now visualise them geometrically.

We will furthermore assume that $\gcd(r_1, r_2) = \gcd(r_1, r_3) = \gcd(r_2, r_3) = \gcd(n_1, n_2, n_3) = 1$.

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This part will be done on the board.

- What does the basis in the general case with four ramified primes look like? What about the module of relations?
- What if we remove the condition of cyclicity of the relative Galois group?
- Is it always true that new Ennola relations only show up in odd dimensions?
- What can be said about the cases of more ramified primes? Will it be possible to explore them by the same means or will we need more advanced machinery?

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