The automorphism tower of a group

Michal Hrbek

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Group automorphisms, center

Let $G = (G, \cdot, ^{-1}, 1)$ be a group. Recall that an automorphism of a group is a bijection $G \to G$ which is a homomorphisms, and that the set of all automorphisms $\mathrm{Aut}(G)$ forms a group, where the operation is their composition - an automorphism group of G.

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Centerless groups

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An easy computation shows that for any $\varphi \in \mathrm{Aut}(G)$ and any $g \in G$ we have:

$$\varphi \circ \iota_g \circ \varphi^{-1} = \iota_{\varphi(g)}.$$

In particular, $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$, and:

Lemma

If G is centerless, then $\mathrm{Aut}(G)$ is centerless too.

The automorphism tower of a centerless group

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$$G \xhookrightarrow{\pi_G} \mathrm{Aut}(G) \xhookrightarrow{\pi_{\mathrm{Aut}(G)}} \mathrm{Aut}(\mathrm{Aut}(G)) \xhookrightarrow{\pi_{\mathrm{Aut}(\mathrm{Aut}(G))}} \cdots$$

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Does it terminate? Denoting the n-th term in this sequence by G_n, is there $n < \omega$ such that $\pi_{G_n} : \mathrm{Aut}(G_n) \to G_n$ is an isomorphism? That is, $\mathrm{Inn}(G_n) = \mathrm{Aut}(G_n)$.

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Theorem (Wielandt (1939))

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$$S_n \xhookrightarrow{\pi_{S_n}} \mathrm{Aut}(S_n) = S_n \hookrightarrow \cdots, n > 6,$$

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But! Let $D_\infty = \langle x, y \mid x^2 = y^2 = 1 \rangle$, then $\mathrm{Aut}(D_\infty) \simeq D_\infty$, but $\mathrm{Inn}(D_\infty) \neq \mathrm{Aut}(D_\infty)$!!

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- If $\alpha + 1$ is succesor, we look at $\pi_{G_\alpha} : G_\alpha \hookrightarrow \mathrm{Aut}(G_\alpha)$. Then we let $G_{\alpha+1}$ be a group isomorphic to $\mathrm{Aut}(G_\alpha)$, which contains G_α as a normal subgroups in a way that:

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  (In other words, we identify G_α with $\mathrm{Inn}(G_\alpha)$.)
- If λ is a limit ordinal, we let $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$. Since $Z(G_\lambda)$, we can continue this way ad nauseum.

We say that the automorphism tower of a centerless group G terminates at step α if $G_\alpha = G_{\alpha+1}$ (and thus $G_\beta = G_\alpha$ for all $\beta > \alpha$). In other words, G_α is a complete group - each automorphism of it is inner.

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The automorphism tower of any centerless group G terminates at some step α, where $\alpha < (2^{|G|})^+$.

Recall that if κ is a cardinal number, then κ^+ denotes its succesor - a smallest cardinal bigger than κ. If you believe in the Generalized Continuum Hypothesis, this boiles down to $\kappa^+ = 2^\kappa$.

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- Numerology: $|G_\alpha| \leq 2^{|G|}$.

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- Dirty trick again: $\iota_{g_\alpha} \circ \iota_{g_\beta}^{-1} \in C_{G_\lambda}(G) = 1$, and so $g_\alpha = g_\beta$ for any $\alpha, \beta \in U$.
- Since U is unbounded, we conclude that $\varphi = \iota_{g_\alpha}$ (for any $\alpha \in U$), and thus is inner.

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- For limit steps, we let

$$G_\lambda = \varinjlim_{\alpha<\lambda} G_\alpha,$$

  the direct limit of the direct system $(G_\alpha, \pi_\alpha \mid \alpha < \lambda)$.

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$$D_8 \to D_8 \to D_8 \to \cdots \to \mathbb{Z}_2 \to 1$$

Example of a countable group G for which $|\mathrm{Aut}(G)| = 2^\omega$, and $|\mathrm{Aut}(\mathrm{Aut}(G))| = (2^\omega)^\omega$!

$$\bigoplus_{p \in \mathbb{P}} \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}^\omega \times \mathbb{Z}_2^\omega \to \mathrm{Aut}(\mathbb{Z}^\omega) \times GL(V).$$

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Proof.

- By Thomas's theorem, it is enough to show that there is λ such that G_λ is centerless.
- By the way direct limits work, we have for each $\alpha < \beta$ a canonical map $\pi_{\alpha,\beta} : G_\alpha \to G_\beta$ satisfying

$$\pi_\alpha = \pi_\beta \circ \pi_{\alpha,\beta},$$

and

$$\pi_{\beta,\gamma} \circ \pi_{\alpha,\beta} = \pi_{\alpha,\gamma}$$

whenever $\alpha < \beta < \gamma$.

□

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- For each ordinal α, define

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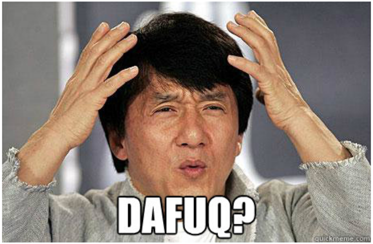

Thanks!