PDE 2, 2022-2023 Written exam June 19, 2023

Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ________________________________

Question	1	2	**Score**
Maximum points	100	100	200
Points			

[100] 1. Let $\Omega := (0,1)^3$ and $T > 0$. Assume that $\{u^n\}_{n=1}^{\infty} \subset \mathcal{C}^1(\overline{(0,T)\times\Omega})$ is a sequence of smooth (w.r.t. x) functions on $(0,T)\times\Omega$ fulfilling for all $f \in L^2(0,T;L^2(\Omega))$ and all $g \in \mathcal{C}_0^{\infty}([0,T];W^{1,\infty}(\Omega))$

$$\lim_{n\to\infty}\int_0^T\int_\Omega u^n f = 0,$$
$$\lim_{n\to\infty}\int_0^T\int_\Omega \frac{\partial u^n}{\partial x_k} f = 0 \qquad \text{for k=1,2,3}$$
$$\int_0^T\int_\Omega u^n \partial_t g = \int_0^T \nabla g\cdot \mathbf{v}^n,$$

where $\mathbf{v}^n$ is bounded in $\in L^{\frac{143}{142}}(0,T;L^{\frac{33}{32}}(\Omega))$.

Decide what from the following is true and prove it / find a counterexample.

1)
$$\lim_{n\to\infty}\int_0^T\int_{\partial\Omega} |u^n|^2 = 0.$$

2)
$$\lim_{n\to\infty}\int_0^T \|u^n\|_{L^8(\Omega)} = 0.$$

Solution:

First two lines tell us that u^n is a sequence such that

$$u^n \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)).$$

Consequently, using the basis from functional analysis, we also have that there exists a constant C such that

$$\sup_{n\in\mathbb{N}}\int_0^T \|u^n\|_{1,2}^2 \le C < \infty. \tag{1}$$

Next, we use also the third line of our information and the Aubin-lions lemma and show that

$$\lim_{n\to\infty}\int_0^T \|u^n\|_2^2 = 0. \tag{2}$$

To do so, we identify $V_1 := W^{1,2}$, $V_2 := L^2$ and $V_3 = (W^{1,33}(\Omega))^*$. Then $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$. We also have the u^n is bounded in $L^2(0,T;V_1)$ and we need to say that $\partial_t u^n$ is bounded at least in $L^1(0,T;V_3)$. In fact we prove more. From the last line it follows that

$$\langle \partial_t u^n, g\rangle = -\int_\Omega \nabla g \cdot \mathbf{v}$$

for almost all $t \in (0,T)$. Therefore, using the density of smooth function in $W^{1,33}$, we have

$$\|\partial_t u^n\|_{V_3} = \sup_{g\in C^1; \|g\|_{1,33}\le 1} -\int_\Omega \nabla g \cdot \mathbf{v}^2 \le \sup_{g\in C^1; \|g\|_{1,33}\le 1} \|\nabla g\|_{33}\|\mathbf{v}^n\|_{\frac{33}{32}} \le \|\mathbf{v}^n\|_{\frac{33}{32}}$$

Therefore, we have

$$\int_0^T \|\partial_t u^n\|^{\frac{143}{142}} \le \int_0^T \|\mathbf{v}^n\|_{\frac{33}{32}}^{\frac{143}{142}} \le C.$$

Thus, we may use the Aubin–Lions lemma and we see that (2)holds.

We claim that 1) holds true. We estimate only the behaviour on a part of $\partial\Omega$, the rest is done similarly. Hence we have

$$\begin{aligned}\int_0^1\int_0^1 |u^n(x,y,0)|^2 dx\,dy &= \int_0^1\int_0^1\int_0^1 \frac{d}{dz}\left((z-1)|u^n(x,y,z)|^2\right) dx\,dy\,dz\\ &= \int_0^1\int_0^1\int_0^1 |u^n(x,y,z)|^2 + 2(z-1)u^n(x,y,z)\frac{\partial u^n(x,y,z)}{\partial z} dx\,dy\,dz\\ &\le \|u^n\|^2_{L^2(\Omega)} + 2\|u^n\|_{L^2(\Omega)}\|\nabla u^n\|_{L^2(\Omega)}.\end{aligned}$$

Integration over time and using again the Hölder inequality, we have

$$\int_0^T \|u^n\|^2_{L^2(\partial\Omega)} \le C(\int_0^T \|u^n\|_2^2 + \left(\int_0^T \|u^n\|_2^2\right)^{\frac12}\left(\int_0^T \|u^n\|_{1,2}^2\right)^{\frac12}$$

Using (1) and (2) we get 1).

Concerning 2), we claim it is not true. We set

$$u^n := \frac{(\ln n)^{-\frac18}}{(n^{-1}+|\mathbf{x}|^2)^{\frac{3}{16}}}$$

Note that u^n is smooth function independent of time. In addition we can set $\mathbf{v}^n = 0$ and then the third assumption is valid. Moreover, we have

$$\|u^n\|_{1,2}^2 \le C(\ln n)^{-\frac18}\int_\Omega \frac{1}{|\mathbf{x}|^{2+\frac34}} \le \tilde{C}(\ln n)^{-\frac18} \to 0$$

as $n \to \infty$. On the other hand

$$\int_\Omega |u^n|^8 \ge c(\ln n)^{-1}\int_0^1 \frac{r^2}{(n^{-1}+r^2)^{\frac32}} \ge \tilde{c}(\ln n)^{-1}\int_0^1 \frac{r^2}{(n^{-\frac32}+r^3)} \ge \overline{c} > 0.$$

[100] 2. Let $\Omega \subset \mathbb{R}^2$ be Lipschitz and $V \subset L^r(\Omega;\mathbb{R}^2)$ defined as

$$V := \{f = (f_1, f_2); f_1, f_2 \in L^r(\Omega),\ \forall u \in \mathcal{C}_0^\infty(\Omega) \int_\Omega f_1 \frac{\partial u}{\partial x_1} + f_2 \frac{\partial u}{\partial x_2}\, dx = 3 \int_\Omega u\, dx\}$$

with some $r \in (1, \infty)$. Consider the problem: Find $f \in V$ such that for all $\tilde{f} \in V$ there holds

$$\int_\Omega (f_1^2 + f_2^2 + f_1 f_2)^{\frac{r}{2}}\, dx \leq \int_\Omega (\tilde{f}_1^2 + \tilde{f}_2^2 + \tilde{f}_1 \tilde{f}_2)^{\frac{r}{2}}\, dx. \qquad (\mathcal{P})$$

30% Show that there exists unique $f \in V$ solving $(\mathcal{P})$

20% Write down the Euler–Lagrange equation corresponding to $(\mathcal{P})$

50% The problem $(\mathcal{P})$ is a dual problem to certain pde. Find this pde a precisely show the correspondence between the minimizer f of $(\mathcal{P})$ and the solution u of founded pde.

Solution:

First, one should observe that the set V consists of $f \in L^r(\Omega;\mathbb{R}^2)$ such that "formally"

$$\text{div } f = -3.$$

Hence defining $\hat{f} := (-3x, 0)$, we see that $\hat{f} \in V$ and so V is not empty. Next, we define a symmetric matrix

$$a_{ij} := \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{2} & \text{if } i \neq l. \end{cases}$$

This matrix is positively definite and therefore we can introduce a scalar product on $\mathbb{R}^2$ as

$$(u, v)_a := \sum_{i,j=1}^{2} a_{ij} u_i u_j \qquad \forall u, v \in \mathbb{R}^2.$$

The problem $\mathcal{P}$ thus be equivalently formulated as: Find $f \in V$ such that for all $\tilde{f} \in V$ there holds

$$\int_\Omega (f, f)_a^{\frac{r}{2}}\, dx \leq \int_\Omega (\tilde{f}, \tilde{f})_a^{\frac{r}{2}}\, dx. \qquad (\mathcal{P}_a)$$

Next, we want to show the existence of a minimizer. The functional $(f, f)_a^{\frac{r}{2}}$ is evidently coercive and since V is not empty, we can find a sequence $\{f^n\}_{n=1}^\infty$ such that

$$f^n \rightharpoonup f \qquad \text{weakly in } L^r(\Omega;\mathbb{R}^2) \qquad (3)$$

and that

$$\inf_{g \in V} \int_\Omega (g, g)_a^{\frac{r}{2}}\, dx = \lim_{n\to\infty} \int_\Omega (f^n, f^n)_a^{\frac{r}{2}}\, dx$$

Consequently, if we show that $f \in V$ and that

$$\lim_{n\to\infty} \int_\Omega (f^n, f^n)_a^{\frac{r}{2}}\, dx \geq \int_\Omega (f, f)_a^{\frac{r}{2}}\, dx \qquad (4)$$

then f will be a minimizer.

To show that $f \in V$, we use that $f^n \in V$, i.e., $\forall u \in \mathcal{C}_0^\infty(\Omega)$ we have that

$$\int_\Omega f_1^n \frac{\partial u}{\partial x_1} + f_2^n \frac{\partial u}{\partial x_2}\, dx = 3 \int_\Omega u\, dx.$$

Next, having (**??**), we can let $n \to \infty$ in the above identity and to show that $f \in V$. To check (4), we use the theorem about the weak lower semicontinuity for which it is enough to prove that

$$(f,f)_a^{\frac{r}{2}}$$

is convex. But since $(\cdot,\cdot)_a$ is a scalar product and $r > 1$, this claim is evident and we have a minimizer. In addition, since it is strictly convex, the minimizer is unique.

Next, we focus on Euler-Lagrange equation. We need to have a competitive functions in the set V and therefore define

$$V_d := \{f = (f_1, f_2); f_1, f_2 \in L^r(\Omega),\ \forall u \in \mathcal{C}_0^\infty(\Omega) \int_\Omega f_1 \frac{\partial u}{\partial x_1} + f_2 \frac{\partial u}{\partial x_2}\, dx = 0\}$$

Then evidently, if $f \in V$ and $h \in V_d$ then for all $\varepsilon > 0$ we have $\tilde{f} := f + \varepsilon h \in V$. We use this setting in $(\mathcal{P}_a)$ to get

$$\begin{aligned}
0 &\le \varepsilon^{-1} \int_\Omega (f+\varepsilon h, f+\varepsilon h)_a^{\frac{r}{2}} - (f,f)_a^{\frac{r}{2}}\, dx \\
&= \varepsilon^{-1} \int_\Omega \int_0^1 \frac{d}{dt} (f+\varepsilon t h, f+\varepsilon t h)_a^{\frac{r}{2}}\, dt\, dx \\
&= \int_\Omega \int_0^1 (f+\varepsilon t h, f+\varepsilon t h)_a^{\frac{r-2}{2}} (f+\varepsilon t h, h)_a\, dt\, dx
\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we can let $\varepsilon \to 0_+$ to conclude that for all $h \in V_d$

$$0 \le \int_\Omega (f,f)_a^{\frac{r-2}{2}} (f,h)_a\, dx$$

Since $\pm h \in V_d$ the above inequality implies that for all $h \in V_d$

$$0 = \int_\Omega (f,f)_a^{\frac{r-2}{2}} (f,h)_a\, dx = \int_\Omega \left(\sum_{i,j=1}^2 a_{ij} f_i f_j \right)^{\frac{r-2}{2}} \sum_{i,j=1}^2 a_{ij} f_i h_j\, dx \tag{E-L}$$

which are the desired Euler–Lagrange equations. Moreover, we have (it follows from the strict monotonicity) that the solution of the Euler–Lagrange equations is unique.

To find a dual problem, let us consider the following: Find $u \in W_0^{1,r'}(\Omega)$ such that for all $v \in W_0^{1,r'}(\Omega)$

$$\int_\Omega \frac{(\nabla u, \nabla u)_{a^{-1}}^{\frac{r'}{2}}}{r'} - 3u\, dx \le \int_\Omega \frac{(\nabla u, \nabla v)_{a^{-1}}^{\frac{r'}{2}}}{r'} - 3v\, dx \tag{D}$$

where a^{-1} denote the inverse matrix to a (which is again positively definite). Using the same arguments as above, we can find a unique minimizer. Also repeating almost step by step the above procedure, we deduce the Euler–Lagrange equation valid for all $v \in W^{1,r'}(\Omega)$

$$0 = \int_\Omega (\nabla u, \nabla u)_{a^{-1}}^{\frac{r'-2}{2}} (\nabla u, \nabla v)_{a^{-1}} - 3v \, dx. \tag{E-L II}$$

Notice that this is just weak formulation of the following problem

$$\begin{aligned} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left((\nabla u, \nabla u)_{a^{-1}}^{\frac{r'-2}{2}} a_{ij}^{-1} \frac{\partial u}{\partial x_i} \right) &= 3 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Finally, we show the relation between ∇u and the minimizer f. Let us define

$$\tilde{f}_j := \sum_{i=1}^{2} (\nabla u, \nabla u)_{a^{-1}}^{\frac{r'-2}{2}} a_{ij}^{-1} \frac{\partial u}{\partial x_i}.$$

Then it follows from (E-L II) that $\tilde{f} \in V$. Finally, we check that $\tilde{f}$ solves (E-L) and due to the uniqueness of a solution to (E-L) wededuce that it is also a minimizer to the original problem. Using a simple algebraic manipulation, we get that

$$\nabla u = (\tilde{f}, \tilde{f})_a^{\frac{r-2}{2}} a\tilde{f}$$

and therefore for every $h \in V_d$

$$\int_\Omega (\tilde{f}, \tilde{f})_a^{\frac{r-2}{2}} (\tilde{f}, h)_a = \int_\Omega \nabla u \cdot h \, dx = 0$$

which follows from the definition of V_d. Hence, $\tilde{f}$ is a minimizer to the original problem.