Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ______________________

Question	1	2	**Score**
Maximum points	100	100	200
Points			

[100] 1. Formulate and prove the Gagliardo-Nirenberg inequality. Formulate and prove the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \in [1, d)$, where $\Omega \subset \mathbb{R}^d$. What embedding holds in case $\Omega = \mathbb{R}^d$?

Solution:

See lecture.

[100] 2. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set and let $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \dots, d$. Moreover assume that there exists $\alpha > 0$ such that for all $z \in \mathbb{R}^d$ and almost all $x \in \Omega$

$$\sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \geq \alpha |z|^2.$$

Define the x-dependent norm in $\mathbb{R}^d$ as

$$|z|^2_{a(x)} := \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j.$$

Next, let $p \in (1, \infty)$ and $f : \Omega \to \mathbb{R}^d$ be given such that $f \in L^p(\Omega; \mathbb{R}^d)$ and consider a set

$$\mathcal{S} := \left\{ g = (g_1, \dots, g_d) \in L^p(\Omega; \mathbb{R}^d);\ \int_\Omega (g + f) \cdot \nabla u \, dx = 0 \text{ for all } u \in W_0^{1,p'}(\Omega) \right\}.$$

40% Show that there exists a unique $g \in \mathcal{S}$ such that

$$\int_\Omega |g(x)|^p_{a(x)} \, dx \leq \int_\Omega |h(x)|^p_{a(x)} \, dx \qquad \text{for all } h \in \mathcal{S}. \tag{Min}$$

10% What is the Euler-Lagrange equation of the above problem?

50% Find an elliptic PDE to which (Min) is a dual problem. Show the relation between g from (Min) and the solution to that PDE.

Solution:

First of all, denoting

$$a^s := \frac{1}{2}(a + a^T), \quad \text{i.e., } a^s_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$$

we see that

$$|z|^2_{a(x)} = |z|^2_{a^s(x)}.$$

Consequently, we consider in what follows only symmetric matrix a, otherwise we take only its symmetric part. Next, we recall some basics facts from linear algebra. Since, a is elliptic and symmetric, we see that

$$(u, v)_a := \sum_{i,j=1}^{d} a_{ij} u_i v_j$$

is a scalar product on $\mathbb{R}^d$. Consequently, we have

$$|(u, v)_a| \leq |u|_a |v|_a. \tag{1}$$

Moreover, we from ellipticity it follows that there exists the inverse elliptic matrix $a^{-1} \in L^\infty(\Omega; \mathbb{R}^{d \times d})$. Finally, we derive an auxiliary inequality (replacing so the argument based on the convexity later)

$$
\begin{aligned}
|u|_a^p - |v|_a^p &= \int_0^1 \frac{d}{dt}|v-t(v-u)|_a^p \, dt = p\int_0^1 |v-t(v-u)|_a^{p-2}(v-t(v-u),u-v)_a \, dt \\
&= p\int_0^1 \left(\frac{d}{dt}(t-1)\right)|v-t(v-u)|_a^{p-2}(v-t(v-u),u-v)_a \, dt.
\end{aligned}
$$

Using integration by parts we continue

$$
\begin{aligned}
&= p|v|_a^{p-2}(v,u-v)_a + p\int_0^1 (1-t)\left(\frac{d}{dt}|v-t(v-u)|_a^{p-2}(v-t(v-u),u-v)_a\right) dt \\
&= p|v|_a^{p-2}(v,u-v)_a + p\int_0^1 \frac{1-t}{|v-t(v-u)|_a^{2-p}}\left(\frac{(p-2)(v-t(v-u),u-v)_a^2}{|v-t(v-u)|_a^2} + |u-v|_a^2\right) dt.
\end{aligned}
$$

Hence, using (1), we see that

$$
|u|_a^p - |v|_a^p \ge p|v|_a^{p-2}(v,u-v)_a + \begin{cases} \displaystyle\int_0^1 \frac{(1-t)|u-v|_a^2}{|v-t(v-u)|_a^{2-p}} \, dt & \text{if } p \ge 2 \\ \displaystyle\int_0^1 \frac{(1-t)(p-1)|u-v|_a^2}{|v-t(v-u)|_a^{2-p}} \, dt & \text{if } p \in (1,2). \end{cases} \tag{Convex}
$$

Next, we focus on the existence of a minimizer. Denoting

$$
I := \inf_{h\in\mathcal{S}} \int_\Omega |h(x)|_{a(x)}^p \, dx,
$$

we deduce from the definition of infima that there exists a sequence $g^n \in \mathcal{S}$ such that

$$
I = \lim_{n\to\infty} \int_\Omega |g^n(x)|_{a(x)}^p \, dx.
$$

On the other hand since $-f \in \mathcal{S}$ we see that

$$
I \le \int_\Omega |f(x)|_{a(x)}^p \, dx \le C\|f\|_p^p
$$

and therefore there exists n_0 such that for all $n \ge n_0$ we have

$$
\alpha\|g^n\|_p^p \le \int_\Omega |g^n(x)|_{a(x)}^p \, dx \le I + 1 \le C(f).
$$

Thus, due to the reflexivity, we see that we can extract a subsequence that we do not relabel such that

$$
g^n \rightharpoonup g \qquad \text{weakly in } L^p(\Omega;\mathbb{R}^d). \tag{2}
$$

Moreover, it is evident from the definition of $\mathcal{S}$ and the above convergence result that

$$
g \in \mathcal{S}.
$$

Thus, to finish the proof of the existence it is sufficient to show (weak-lower semicontinuity) that

$$\lim_{n\to\infty} \int_\Omega |g^n(x)|^p_{a(x)} - |g(x)|^p_{a(x)} \, dx \geq 0. \tag{WLS}$$

Using (Convex) with $u := g^n$ and $v := g$, we gain

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega |g^n(x)|^p_{a(x)} - |g(x)|^p_{a(x)} \, dx &\geq p \lim_{n\to\infty} \int_\Omega |g(x)|^{p-2}_{a(x)} \left(g(x), g^n(x) - g(x)\right)_{a(x)} \, dx \\
&= p \lim_{n\to\infty} \sum_{i,j=1}^d \int_\Omega \underbrace{|g(x)|^{p-2}_{a(x)} a_{ij}(x) g_i(x)}_{\in L^{p'}} \underbrace{(g^n_j(x) - g_j(x))}_{\rightharpoonup 0 \text{ in } L^p} \, dx \\
&= 0,
\end{aligned}$$

where the last equality follows from (2). Thus, the proof of the existence is complete.

Next, we derive the Euler-Lagrange equations. Thus, for arbitrary $w \in L^p(\Omega; \mathbb{R}^d)$ such that

$$\int_\Omega w \cdot \nabla u \, dx = 0 \qquad \text{for all } u \in W_0^{1,p'}(\Omega) \tag{3}$$

we see that for arbitrary $t > 0$, we have $h := g + tw \in \mathcal{S}$. Thus, using such an h in (Min) we get

$$\int_\Omega |g(x)|^p_{a(x)} \, dx \leq \int_\Omega |g(x) + tw(x)|^p_{a(x)} \, dx.$$

Hence, dividing by t and letting $t \to 0_+$ we deduce

$$0 \leq \lim_{t\to 0_+} \int_\Omega \frac{|g(x) + tw(x)|^p_{a(x)} - |g(x)|^p_{a(x)}}{t} \, dx = p \int_\Omega |g(x)|^{p-2}_{a(x)} (g(x), w(x))_{a(x)} \, dx.$$

Since $-w$ is also a possible setting, we see that the Euler-Lagrange equatins is of the form

$$0 = \int_\Omega |g(x)|^{p-2}_{a(x)} (g(x), w(x))_{a(x)} \, dx \tag{E-L}$$

for all $w \in L^p(\Omega; \mathbb{R}^d)$ fulfilling

$$\int_\Omega w \cdot \nabla u \, dx = 0 \qquad \text{for all } u \in W_0^{1,p'}(\Omega).$$

Concerning uniqueness, let $g^1 \neq g^2$ be two minimizers. Then, it follows from (Convex) that

$$\int_\Omega |g^1(x)|^p_{a(x)} - |g^2(x)|^p_{a(x)} \, dx > p \int_\Omega |g^2(x)|^{p-2}_{a(x)} (g^2(x), g^1(x) - g^2(x))_{a(x)} \, dx, \tag{4}$$

where the strict inequality follows from the fact that the second term on the right hand side of (Convex) is surely positive on a set of nonzero measure due to the assumption $g^1 \neq g^2$. Since g^1, g^2 are minimizers, we know that

$$I = \int_\Omega |g^1(x)|^p_{a(x)} \, dx = \int_\Omega |g^2(x)|^p_{a(x)} \, dx$$

and consequently the left hand side of (4) is zero. On the other hand, since $g^1, g^2 \in \mathcal{S}$ we get

$$\int_\Omega (g^1 - g^2) \cdot \nabla u \, dx = 0 \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

Hence, using (E-L) with $g := g^1$ and $w := g^1 - g^2$, we see that the right hand side of (4) is also zero, which is a contradiction.

For the second part, we follow the lecture and the only reasonable chance for some u being a solution (which up to now we do not know exists) to the primary problem is that it is given by

$$|g(x)|_{a(x)}^{p-2} \sum_{j=1}^{d} a_{ij}(x) g_j(x) = \frac{\partial u}{\partial x_i} \tag{5}$$

Our first goal is to invert (5). Thus, taking the scalar product with g we have

$$|g(x)|_{a(x)}^{p} = (\nabla u(x), g(x)). \tag{6}$$

Next, multiplying (5) by a_{ik}^{-1} and summing with respect to i we get

$$|g(x)|_{a(x)}^{p-2} g_k(x) = \sum_{i=1}^{d} a_{ik}^{-1}(x) \frac{\partial u(x)}{\partial x_i} \tag{7}$$

Thus, taking the scalar product with ∇u we see that

$$|g(x)|_{a(x)}^{p-2} (g(x), \nabla u(x)) = |\nabla u(x)|_{a^{-1}(x)}^{2}. \tag{8}$$

Hence, comparing (6) and (8) we deduce

$$|g(x)|_{a(x)} = |\nabla u(x)|_{a^{-1}(x)}^{\frac{1}{p-1}}, \tag{9}$$

which substituting into (7) leads to

$$g_k(x) = \sum_{i=1}^{d} a_{ik}^{-1}(x) |\nabla u(x)|_{a^{-1}(x)}^{p'-2} \frac{\partial u(x)}{\partial x_i} \tag{10}$$

Hence, taking the scalar product with ∇v, where $v \in W_0^{1,p'}(\Omega)$, we get

$$\int_\Omega |\nabla u(x)|_{a^{-1}(x)}^{p'-2} (\nabla u(x), \nabla v)_{a^{-1}(x)} \, dx = \int_\Omega g(x) \cdot \nabla v(x) \, dx \overset{g \in \mathcal{S}}{=} \int_\Omega f \cdot \nabla v \, dx, \tag{11}$$

which is nothing else than the weak formulation of the problem:

$$\begin{aligned} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(|\nabla u(x)|_{a^{-1}(x)}^{p'-2} a_{ij}^{-1}(x) \frac{\partial u(x)}{\partial x_j} \right) &= \sum_{i=1}^{d} \frac{\partial f(x)}{\partial x_i}, \quad && \text{in } \Omega, \\ u &= u_d && \text{on } \partial\Omega, \end{aligned}$$

where u_d is given Dirichlet boundary condition, e.g., $u_d = 0$. Hence, it remains to prove the existence of $u \in W_0^{1,p'}(\Omega)$ solving (11) for all $v \in W_0^{1,p'}(\Omega)$. We can proceed by using the monotone operator theory, however, since we have already proved the weak lower semicontinuity for certain functional, we proceed differently. In fact we show that to solve (11) is equivalent to find $u \in W_0^{1,p'}(\Omega)$ solving for all $v \in W_0^{1,p'}(\Omega)$

$$\int_\Omega \frac{|\nabla u(x)|^{p'}_{a^{-1}(x)}}{p'} - f(x) \cdot \nabla u(x)\, dx \le \int_\Omega \frac{|\nabla v(x)|^{p'}_{a^{-1}(x)}}{p'} - f(x) \cdot \nabla v(x)\, dx. \tag{P-F}$$

But we have already use the weak lower semicontinuity and therefore the minimum surely exists and in addition is unique. It only remains to show that (P-F) implies (in fact is equivalent to) (11). To do so, we again derive the Euler-Lagrange equation, i.e., we set $v := u + tw$ in (P-F) with arbitrary $w \in W_0^{1,p'}(\Omega)$, divide by t and let $t \to 0_+$ to gain

$$\begin{aligned} \int_\Omega f(x) \cdot \nabla w(x)\, dx &\le \lim_{t\to 0_+} \frac{1}{p'} \int_\Omega \frac{|\nabla(u + tw(x)|^{p'}_{a^{-1}(x)} - |\nabla u(x)|^{p'}_{a^{-1}(x)}}{t}\, dx \\ &= \int_\Omega |\nabla u|^{p'-2}_{a^{-1}(x)} (\nabla u(x), \nabla w(x))_{a^{-1}(x)}\, dx, \end{aligned} \tag{P-F}$$

which directly implies (11).