Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ______________________________

Question	1	2	**Score**
Maximum points	100	100	200
Points			

[100] 1. Formulate and prove the Aubin–Lions.

Solution:

See lecture.

[100] 2. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set. Consider the problem: for given $p, q \in (1, \infty)$ and given $\mathbf{f} = (f_1, \dots, f_L) : \Omega \to \mathbb{R}^L$, where $L \in \mathbb{N}$, find $\mathbf{u} = (u_1, \dots, u_L) : \Omega \to \mathbb{R}^L$ solving for all $\nu = 1, \dots, L$

$$
\begin{aligned}
-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(|\nabla \mathbf{u}|^{p-2}\frac{\partial u_\nu}{\partial x_i}\right) + |\mathbf{u}|^{q-2}u_\nu = f_\nu \quad &\text{in } \Omega,\\
u_\nu = 0 \quad &\text{on } \partial\Omega,
\end{aligned}
$$

where

$$
\begin{aligned}
|\nabla \mathbf{u}|^2 &:= \sum_{i=1}^{d}\sum_{\nu=1}^{L}\left|\frac{\partial u_\nu}{\partial x_i}\right|^2,\\
|\mathbf{u}|^2 &:= \sum_{\nu=1}^{L}(u_\nu)^2.
\end{aligned}
$$

For $\mathbf{f} \in L^{q'}(\Omega; \mathbb{R}^L)$ define the notion of a weak solution. Check that for given data such a definition is meaningful. Prove the existence and uniqueness of the weak solution. (*Help: As a leading function space consider* $V := W_0^{1,p}(\Omega; \mathbb{R}^L) \cap L^q(\Omega; \mathbb{R}^L)$)

Solution:

First, we recall some facts proved in the lectures. The mapping $A : \mathbb{R}^{d\times L} \to \mathbb{R}^{d\times L}$ defined as

$$A(\eta) := |\eta|^{p-2}\eta,$$

is strictly monotone p-coercive and has $(p-1)$-growth. Moreover, the mapping $F_1 : \mathbb{R}^{d\times L} \to \mathbb{R}$ defined as

$$F_1(\eta) := \frac{|\eta|^p}{p},$$

is strictly convex, p-coercive and has p-growth and fulfills

$$\frac{\partial F_1(\eta)}{\partial \eta} = A(\eta).$$

Similarly, the mapping $B : \mathbb{R}^L \to \mathbb{R}^L$ defined as

$$B(\mathbf{u}) := |\mathbf{u}|^{q-2}\mathbf{u},$$

is strictly monotone, q-coercive and has $(q-1)$-growth. Moreover, the mapping $F_2 : \mathbb{R}^L \to \mathbb{R}$ defined as

$$F_2(\mathbf{u}) := \frac{|\mathbf{u}|^q}{q},$$

is strictly convex, q-coercive, has q-growth and fulfills

$$\frac{\partial F_2(\mathbf{u})}{\partial \mathbf{u}} = B(\mathbf{u}).$$

The problem can be written in the form

$$
\begin{aligned}
-\operatorname{div} A(\nabla \mathbf{u}) + B(\mathbf{u}) = \mathbf{f} \quad &\text{in } \Omega,\\
\mathbf{u} = 0 \quad &\text{on } \partial\Omega.
\end{aligned}
\tag{*}
$$

Formal derivation of the notion of weak solution: Let $\mathbf{v} \in V$ be arbitrary and take the scalar product of (*) and $\mathbf{v}$, i.e., multiply the ν-the equation in (*) by v_ν and sum the result over $\nu = 1, \ldots, L$. Finally integrate over Ω to get

$$-\int_\Omega \operatorname{div} A(\nabla \mathbf{u}) \cdot \mathbf{v} \, dx + \int_\Omega B(\mathbf{u}) \cdot \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx.$$

For the first integral use integration by parts and the fact that $\mathbf{v} = 0$ on $\partial\Omega$ (so the boundary term formally vanishes) to get the weak formulation

$$\int_\Omega A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} \, dx + \int_\Omega B(\mathbf{u}) \cdot \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \qquad \text{for all } \mathbf{v} \in V. \tag{**}$$

We check that all integrals are well defined for $\mathbf{u}, \mathbf{v} \in V$. Using the Hölder inequality we have

$$\begin{aligned}
\left| \int_\Omega A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} \, dx \right| &\le \int_\Omega |A(\nabla \mathbf{u})||\nabla \mathbf{v}| \, dx \le \int_\Omega |\nabla \mathbf{u}|^{p-1} |\nabla \mathbf{v}| \, dx \le \|\mathbf{u}\|_p^{p-1} \|\nabla \mathbf{v}\|_p < \infty, \\
\left| \int_\Omega B(\mathbf{u}) \cdot \mathbf{v} \, dx \right| &\le \int_\Omega |B(\mathbf{u})||\mathbf{v}| \, dx \le \int_\Omega |\mathbf{u}|^{q-1} |\mathbf{v}| \, dx \le \|\mathbf{u}\|_q^{q-1} \|\mathbf{v}\|_q < \infty, \\
\left| \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \right| &\le \int_\Omega |\mathbf{f}||\mathbf{v}| \, dx \le \|\mathbf{f}\|_{q'} \|\mathbf{v}\|_q < \infty
\end{aligned}$$

so (**) is meaningful.

Uniqueness of the weak solution: Let $\mathbf{u}_1, \mathbf{u}_2 \in V$ be two weak solutions to (**). Subtracting the weak formulation for $\mathbf{u}_2$ from that one for $\mathbf{u}_1$ we deduce that

$$\int_\Omega (A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot \nabla \mathbf{v} \, dx + \int_\Omega (B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot \mathbf{v} \, dx = 0 \qquad \text{for all } \mathbf{v} \in V.$$

Hence, setting $\mathbf{v} := \mathbf{u}_1 - \mathbf{u}_2$ (which is a possible choice since $\mathbf{u}_1 - \mathbf{u}_2 \in V$), we have

$$\int_\Omega (A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) \, dx + \int_\Omega (B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, dx = 0.$$

Using the monotonicity of A and of B, it directly follows that

$$\begin{aligned}
(A(\nabla \mathbf{u}_1) - A(\nabla \mathbf{u}_2)) \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2) &\equiv 0 \quad \text{a.e. in } \Omega, \\
(B(\mathbf{u}_1) - B(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) &\equiv 0 \quad \text{a.e. in } \Omega.
\end{aligned}$$

Consequently, using the strict monotonicity of B we get that $\mathbf{u}_1 = \mathbf{u}_2$ almost everywhere in Ω.

Existence of a weak solution via variational approach: Look for $\mathbf{u} \in V$ solving

$$\int_\Omega F_1(\nabla \mathbf{u}) + F_2(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \, dx \le \int_\Omega F_1(\nabla \mathbf{w}) + F_2(\mathbf{w}) - \mathbf{f} \cdot \mathbf{w}, \text{ for all } \mathbf{w} \in V. \tag{***}$$

First, assume that such $\mathbf{u}$ exists. Then setting $\mathbf{w} := \mathbf{u} + t\mathbf{w}$ with $t > 0$ we have

$$0 \le \int_\Omega \frac{F_1(\nabla \mathbf{u} + t\nabla \mathbf{v}) - F_1(\nabla \mathbf{u})}{t} + \frac{F_2(\mathbf{u} + t\mathbf{v}) - F_2(\mathbf{u})}{t} - \mathbf{f} \cdot \mathbf{v} \, dx, \text{ for all } \mathbf{v} \in V.$$

Thus, letting $t \to 0_+$, using the properties of F_1, F_2, A and B and using the results proved in the lectures we get

$$0 \le \int_\Omega A(\nabla \mathbf{u}) \cdot \nabla \mathbf{v} + B(\mathbf{u}) \cdot \mathbf{v} - \mathbf{f} \cdot \mathbf{v} \, dx, \text{ for all } \mathbf{v} \in V.$$

Since we can set also $-\mathbf{v}$ above, we directly have (**).

To prove the existence of $\mathbf{u}$ solving (***), we first define

$$I := \inf_{\mathbf{w} \in V} \int_\Omega F_1(\nabla \mathbf{w}) + F_2(\mathbf{w}) - \mathbf{f} \cdot \mathbf{w} \, dx \le 0,$$

where the last inequality follows from the fact that $0 \in V$. From the definition of inf there must exist a sequence $\{\mathbf{u}^n\}_{n=1}^\infty$ such that

$$I = \lim_{n \to \infty} \int_\Omega F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \le 0.$$

Consequently, there must exist n_0 such that for all $n \ge n_0$

$$\int_\Omega F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \le 1,$$

from which it follows that

$$\frac{\|\nabla \mathbf{u}^n\|_p^p}{p} + \frac{\|\nabla \mathbf{u}^n\|_q^q}{q} \le 1 + \int_\Omega \mathbf{f} \cdot \mathbf{u}^n \, dx \le 1 + \|\mathbf{f}\|_{q'} \|\mathbf{u}^n\|_q \le \frac{\|\mathbf{u}^n\|_q^q}{2q} + 1 + 2^{q'} \|\mathbf{f}\|_{q'}^{q'}.$$

Finally, using the Poincaré inequality (note that $\mathbf{u}$ has zero trace) we get that

$$\|\mathbf{u}\|_V \le C(\Omega, \mathbf{f}).$$

Due to the reflexivity, we can extract a non-relabeled subsequence and find $\mathbf{u} \in V$ such that

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \text{ weakly in } V.$$

In particular, we have that

$$\begin{aligned} \nabla \mathbf{u}^n &\rightharpoonup \nabla \mathbf{u} \quad \text{weakly in } L^p(\Omega; \mathbb{R}^{d \times L}), \\ \mathbf{u} &\rightharpoonup \mathbf{u} \qquad \text{weakly in } L^q(\Omega; \mathbb{R}^L). \end{aligned}$$

Finally, we use the weak-lower semicontinuity of convex functionals (proved in the lectures and here applied to F_1 since it has p-growth and A has $(p-1)$-growth and to F_2 since it has q growth and B has $(q-1)$ growth) to get

$$I = \lim_{n \to \infty} \int_\Omega F_1(\nabla \mathbf{u}^n) + F_2(\mathbf{u}^n) - \mathbf{f} \cdot \mathbf{u}^n \, dx \ge \int_\Omega F_1(\nabla \mathbf{u}) + F_2(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \, dx \ge I,$$

where the last inequality follows from the definition of I. Hence, we see that $\mathbf{u}$ satisfies (***).

Existence via monotone operator theory: Since V is a separable space there exists a countable set $\{\mathbf{w}_i\}_{i=1}^{\infty}$ which is dense in V. We look for the Galerkin approximation $\mathbf{u}^n$ being of the form

$$\mathbf{u}^n := \sum_{i=1}^{n} c_i^n \mathbf{w}_i$$

and solving

$$\int_\Omega A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{w}_i \, dx + \int_\Omega B(\mathbf{u}^n) \cdot \mathbf{w}_i \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w}_i \, dx \qquad \text{for all } i = 1.\dots,n. \tag{**n}$$

Due to the continuity of A and B and their coercivity, such u^n exists (see the lecture). Then multiplying the i-th equation in (**n) by c_i^n and summing the result over $i = 1,\dots,n$ we get

$$\int_\Omega A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_\Omega B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{u}^n \, dx. \tag{En}$$

From this we get

$$\|\mathbf{u}^n\|_V \le C$$

and consequently, using the $(p-1)$-growth of A and $(q-1)$-growth of B also that

$$\|A(\nabla \mathbf{u}^n)\|_{L^{p'}(\Omega;\mathbb{R}^{d\times L})} + \|B(u^n)\|_{L^{q'}(\Omega;\mathbb{R}^L)} \le C$$

and due to the reflexivity of leading spaces (here we use that $p, q \in (1,\infty)$) we have

$$\begin{aligned} \mathbf{u}^n &\rightharpoonup \mathbf{u} && \text{weakly in } V,\\ A(\nabla \mathbf{u}^n) &\rightharpoonup \overline{A} && \text{weakly in } L^{p'}(\Omega;\mathbb{R}^{d\times L}),\\ B(\mathbf{u}^n) &\rightharpoonup \overline{B} && \text{weakly in } L^{q'}(\Omega;\mathbb{R}^{L}). \end{aligned}$$

Then for fix i it is easy to let $n \to \infty$ in (**n) to observe

$$\int_\Omega \overline{A} \cdot \nabla \mathbf{w}_i \, dx + \int_\Omega \overline{B} \cdot \mathbf{w}_i \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w}_i \, dx \qquad \text{for all } i = 1.\dots,\infty.$$

and due tot he density of $\{\mathbf{w}_i\}$ in V also that

$$\int_\Omega \overline{A} \cdot \nabla \mathbf{w} \, dx + \int_\Omega \overline{B} \cdot \mathbf{w} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{w} \, dx \qquad \text{for all } \mathbf{w} \in V. \tag{****}$$

Hence to prove (**) it remains to show that the left hand side of (****) is equal to the left hand side of (**). To prove it, we first set $\mathbf{w} := \mathbf{u}$ in (****) to get

$$\int_\Omega \overline{A} \cdot \nabla \mathbf{u} \, dx + \int_\Omega \overline{B} \cdot \mathbf{u} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx. \tag{E}$$

Then taking the limit in (En) and using (E) we have

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_\Omega B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx = \lim_{n\to\infty} \int_\Omega \mathbf{f} \cdot \mathbf{u}^n \, dx \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx = \int_\Omega \overline{A} \cdot \nabla \mathbf{u} \, dx + \int_\Omega \overline{B} \cdot \mathbf{u} \, dx. \end{aligned} \tag{EE}$$

Finally, for any $\mathbf{w} \in V$, we can use the monotonicity of the operators, (EE) and weak convergence results established above so that we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \int_\Omega (A(\nabla \mathbf{u}^n) - A(\nabla \mathbf{w})) \cdot (\nabla \mathbf{u}^n - \nabla \mathbf{w}) + (B(\mathbf{u}^n) - B(\mathbf{w})) \cdot (\mathbf{u}^n - \mathbf{w})\, dx \\
&= \lim_{n\to\infty} \int_\Omega A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \, dx + \int_\Omega B(\mathbf{u}^n) \cdot \mathbf{u}^n \, dx \\
&\quad - \lim_{n\to\infty} \int_\Omega A(\nabla \mathbf{w}) \cdot (\nabla \mathbf{u}^n - \nabla \mathbf{w}) + B(\mathbf{w}) \cdot (\mathbf{u}^n - \mathbf{w})\, dx \\
&\quad - \lim_{n\to\infty} \int_\Omega A(\nabla \mathbf{u}^n) \cdot \nabla \mathbf{w} + B(\mathbf{u}^n)\mathbf{w}\, dx \\
&= \int_\Omega \overline{A} \cdot \nabla \mathbf{u}\, dx + \int_\Omega \overline{B} \cdot \mathbf{u}\, dx \\
&\quad - \int_\Omega A(\nabla \mathbf{w}) \cdot (\nabla \mathbf{u} - \nabla \mathbf{w}) + B(\mathbf{w}) \cdot (\mathbf{u} - \mathbf{w})\, dx \\
&\quad - \int_\Omega A(\nabla \mathbf{u}) \cdot \nabla \mathbf{w} + B(\mathbf{u})\mathbf{w}\, dx \\
&= \int_\Omega (\overline{A} - A(\nabla \mathbf{w})) \cdot (\nabla \mathbf{u} - \nabla \mathbf{w}) + (\overline{B} - B(\mathbf{w})) \cdot (\mathbf{u} - \mathbf{w})\, dx.
\end{aligned}$$

The Minty trick then finishes the proof.