Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ____________________

Question	1	2	**Score**
Maximum points	100	100	200
Points			

[100] 1. Formulate and prove the Morey lemma. Formulate and prove the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $p \in (d, \infty)$, where $\Omega \subset \mathbb{R}^d$.

Solution:

See lecture.

[100] 2. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set $u_D \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Further, let $g, h : \mathbb{R} \to [-1, 1]$ be continuous functions. Consider the problem: for given data find $u : \Omega \to \mathbb{R}$ solving

$$-\triangle u + g(x_1)\frac{\partial^2 u}{\partial x_1 \partial x_2} + h(u) = f \qquad \text{in } \Omega,$$
$$u = u_D \quad \text{on } \partial\Omega.$$

20% Find a proper definition of a weak solution. Check that for given data such a definition is meaningful.

40% Show the existence of a weak solution.

20% In case that h is nondecreasing show also its uniqueness.

20% Moreover, in case $h \in \mathcal{C}^1$, show that if $h' \geq -\gamma > -\frac{\mu_1}{2}$, where μ_1 is the smallest eigenvalue of the Laplace operator subjected to Ω and zero Dirichlet data, then the solution is also unique.

Solution:

First, we formally derive the weak formulation. Let $v \in W_0^{1,2}(\Omega)$ be arbitrary. We multiply the equation by v and integrate over Ω. For the first two terms, we use the integration by parts and since $v = 0$ on $\partial\Omega$ we obtain

$$\int_\Omega \nabla u \cdot \nabla v - g(x_1)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2} + h(u)v \, dx = \int_\Omega fv \, dx. \tag{1}$$

It is **important** here that in the second term we integrate by parts with respect to x_2 because $g(x_1)$ may not have derivative with respect to x_1 as it is assumed to be only continuous!

We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff (1) holds for all $v \in W_0^{1,2}(\Omega)$ and $u = u_D$ on $\partial\Omega$ in the sense of traces. The fact that all integrals in (1) are finite follows from the Hölder inequality and boundedness of g and h.

Next, we focus on solvability. First, we rewrite the leading term into a bilinear form. Denoting

$$a_{ij}(x) := \begin{cases} 1 & \text{for } i = j, \\ -g(x_1) & \text{for } i = 2, j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

we see that (1) is equivalent to

$$B(u, v) + \int_\Omega h(u)v \, dx = \int_\Omega fv \, dx \qquad \text{for all } v \in W_0^{1,2}(\Omega), \tag{2}$$

where

$$B(u, v) := \int_\Omega \sum_{i,j=1}^d a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} \, dx.$$

Since, $|g| \leq 1$, we gain that $|a_{ij}(x)| \leq 1$ and more importantly

$$\sum_{i,j=1}^d a_{ij}(x) z_i z_j \geq \frac{|z|^2}{2}. \tag{3}$$

Next, we "decide" how to solve (2). By process of elimination, we end up with the Galerkin method. Indeed since the equation is not linear, there is no hope for the Lax-Milgram theorem. Moreover, since h is only continuous and not Lipschitz, there is no hope for non-linear version of the Lax-Milgram theorem. Finally, since the matrix a is not symmetric there is no hope for potential and for using the variational approach. On the other hand the presence of $h(u)$ is only a compact "perturbation" therefore we choose the Galerkin method.

As usual, we look for u being of the form $u = u_D + u_0$, where $u_0 \in W_0^{1,2}(\Omega)$. With this notation, the identity (2) is tantamount to

$$B(u_0, v) + \int_\Omega h(u_0 + u_D)v \, dx = \int_\Omega fv \, dx - B(u_D, v) \qquad \text{for all } v \in W_0^{1,2}(\Omega). \tag{4}$$

Set $V := W_0^{1,2}(\Omega)$ and let $\{w_i\}_{i\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ be linearly independent with a dense span in V. We look for $u_0^N(x) = \sum_{i=1}^N c_i^N w_i(x)$, $c_i^N \in \mathbb{R}$, solving an approximate problem

$$B(u_0^N, w_i) + \int_\Omega (h(u_0^N + u_D) - f)w_i \, dx + B(u_D, w_i) = 0 \qquad \text{for all } i = 1, \dots, N. \tag{5}$$

We know how to attack (5): Let us define $F : \mathbb{R}^N \to \mathbb{R}^N$ as $F(\mathbf{c}^N)_i$ is LHS of $(5)_i$. The function F is clearly continuous and so existence of its zero point is guaranteed by $F(\rho)\cdot\rho \geq 0$ on some (non-degenerate) sphere. First, since the matrix $a(x)$ is elliptic (see (3)) we can use the Poincaré inequality to conclude that the bilinear form B is V-elliptic and V-bounded. Therefore (for some $\alpha > 0$)

$$\begin{aligned}
F(\mathbf{c}^N)\cdot\mathbf{c}^N &= B(u_0^N, u_0^N) + B(u_D, u_0^N) + \int_\Omega (h(u^N + u_D) - f)u_0^N \, dx \\
&\geq \alpha\|u_0^N\|_V^2 - C\|u_0^N\|_V\|u_D\|_V - C\|h(u_0^n + u_D) - f\|_2\|u_0^N\|_V \\
&\geq \frac{\alpha}{2}\|u_0^N\|_V^2 - C^2\|u_D\|_V^2 - C^2\|h(u_0^n + u_D) - f\|_2^2 \\
&\geq \frac{\alpha}{2}\|u_0^N\|_V^2 - C^2\|u_D\|_V^2 - C^2\||f| + 1\|_2^2 \geq \frac{\alpha}{2}\|u_0^N\|_V^2 - K(u_D, f, \Omega),
\end{aligned}$$

where we used the Young inequality repeatedly and the constant K is some generic constant depending only on the data. Since linear independence of $\{w_i\}_{i\in\mathbb{N}}$ ensures $\|u_0^N\|_V \sim |\mathbf{c}^N|$, we are done with proving existence of a zero point for F and therefore a solution u^N to (5). Moreover, since $\mathbf{c}^N$ is zero point of F it follows from the computation above that

$$\frac{\alpha}{2}\|u_0^N\|_V^2 \leq K(u_D, f, \Omega),$$

which is independent of N. Due to the reflexivity of V, we can find a subsequence (that we do not relabel) and $u_0 \in V$ such that

$$u_0^N \rightharpoonup u_0 \qquad \text{weakly in } V.$$

The compact embedding further entails (for a subsequence)

$$u_0^N \to u_0 \qquad \text{strongly in } L^2(\Omega).$$

Finally, the above convergence results implies (for a subsequence)

$$u_0^N \to u_0 \qquad \text{almost everywhere in } \Omega.$$

Hence for fix $i \in \mathbb{N}$ we have no problem to let $N \to \infty$ in (5) (recall h is bounded, so we may invoke the Lebesgie dominated convergence theorem, and B is bilinear, so the weak convergence suffices to pass to the limit), to obtain

$$B(u_0, w_i) + \int_\Omega (h(u_0 + u_D) - f) w_i \, dx + B(u_D, w_i) = 0 \qquad \text{for all } i \in \mathbb{N}.$$

The density property of $\{w_i\}_{i \in \mathbb{N}}$ finally gives (4), so the proof of the existence is finished.
Let now h be non-decreasing and u_1, u_2 be two weak solutions. Subtracting (2) pertinent to u_1 and u_2 and setting $v := u_1 - u_2 \in W_0^{1,2}(\Omega)$, we have

$$B(u_1 - u_2, u_1 - u_2) + \int_\Omega (h(u_1) - h(u_2)(u_1 - u_2) \, dx = 0. \tag{6}$$

The non-decreasing nature of h implies $\big(h(u_1) - h(u_2)\big)(u_1 - u_2) \geq 0$ and so we are left practically with $\|\nabla(u_1 - u_2)\|_2 \leq 0$, meaning $u_1 - u_2 \equiv 0$, as $u_1 - u_2 \in W_0^{1,2}(\Omega)$.
The starting point for the last question will be (6). Notice again that from (3) it follows that

$$B(u_1 - u_2, u_1 - u_2) \geq \frac{1}{2} \|\nabla(u_1 - u_2)\|_2^2$$

and due to additional properties of h also

$$\begin{aligned} \big(h(u_1) - h(u_2)\big)(u_1 - u_2) &= \int_0^1 \frac{d}{dt} h(u_2 + t(u_1 - u_2)) \, dt \, (u_1 - u_2) \\ &= \int_0^1 h'(u_2 + t(u_1 - u_2)) \, dt \, |u_1 - u_2|^2 \\ &\geq -\gamma |u_1 - u_2|^2. \end{aligned}$$

Hence assuming that $u_1 \neq u_2$, we deduce form (6) and form the assumption on γ that

$$\|\nabla(u_1 - u_2)\|_2^2 \leq 2\gamma \|u_1 - u_2\|_2^2 < \mu_1 \|u_1 - u_2\|_2^2.$$

Hence we make a hypothesis that would clearly lead to our objective as long as it held:

$$\|\nabla v\|_2^2 \geq \mu_1 \|v\|_2^2 \quad \text{for any } v \in W_0^{1,2}(\Omega). \tag{7}$$

Let $v \in W_0^{1,2}(\Omega)$. We know there is a sequence $\{v_i\}_{i \in \mathbb{N}} \subset W_0^{1,2}(\Omega)$ and $0 < \mu_1 \leq \mu_2 \leq \ldots \to \infty$ such that for any $i \in \mathbb{N}$ the Laplace equation

$$\begin{aligned} -\Delta v_i &= \mu_i v_i && \text{in } \Omega, \\ v_i &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{8}$$

is satisfied (at least) in the weak sense, $\{v_i\}$ is an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $W_0^{1,2}(\Omega)$. In addition due to the orthogonality and (8) we also have

$$\|\nabla v_i\|_2^2 = \mu_i \|v_i\|_2^2 = \mu_i, \qquad \int_\Omega \nabla v_i \cdot \nabla v_j \, dx = 0 \text{ for } i \neq j. \tag{9}$$

Since $\{v_i\}$ forms a basis we can write

$$v = \sum_{i=1}^{\infty} a_i v_i,$$

where the sum converges in V. Since $\{v_i\}$ is orthonormal in $L^2(\Omega)$ we can use the Parseval equality and therefore

$$\|v\|_2^2 = \sum_{i=1}^{\infty} a_i^2.$$

In addition

$$\begin{aligned}\|\nabla v\|_2^2 &= \lim_{n\to\infty} \left\|\nabla \sum_{i=1}^{n} a_i v_i\right\|_2^2 = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i^2 \|\nabla v_i\|_2^2 + \sum_{i\neq j}^{n} a_i a_j \int_\Omega \nabla v_i \cdot \nabla v_j \; dx \right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i^2 \mu_i = \sum_{i=1}^{\infty} a_i^2 \mu_i,\end{aligned}$$

where for the third equality we used (9). Consequently since $\mu_i \le \mu_j$ for $i \le j$ we gain

$$\|\nabla v\|_2^2 = \sum_{i=1}^{\infty} a_i^2 \mu_i \ge \mu_1 \sum_{i=1}^{\infty} a_i^2 = \mu_1 \|v\|_2^2,$$

which is nothing else than (7). The proof is complete.

Alternative proof of the existence of a solution for real fans of the Lax-Milgram theorem: Let us define the mapping $F : L^2(\Omega) \to L^2(\Omega)$ as $v \mapsto u_0$, where $u_0 \in W_0^{1,2}$ solves the problem

$$B(u_0, v) = \int_\Omega f v \; dx - B(u_D, v) - \int_\Omega h(v + u_D) v \; dx \qquad \text{for all } v \in W_0^{1,2}(\Omega). \tag{10}$$

Since B is bilinear, V-bounded and V-elliptic, and h is bounded we can use the Lax-Milgram theorem to get that for all $v \in L^2(\Omega)$ there exists unique $u_0 \in V$ solving (10). Moreover, due to the continuity of h and the V-ellipticity of B it follows that F is continuous. In addition (again from the Lax-Milgram theorem) we know that

$$\|u_0\| \le C(\|f\|_2 + \|u_D\|_V + \|g(v + u_D)\|_2) \le K,$$

where the constant K does not depend on v since g is bounded function. Therefore, we see that F is continuous mapping that maps $L^2(\Omega)$ into a bounded ball in V. Since $V \hookrightarrow\hookrightarrow L^2(\Omega)$ (the compact embedding of the Sobolev spaces), we see that F is continuous compact mapping that maps $L^2(\Omega)$ into a bounded ball of $L^2(\Omega)$. Therefore it must have a fixed point $\bar{u}_0$. However, using (10) this fixed point must satisfy

$$B(\bar{u}_0, v) + \int_\Omega h(\bar{u}_0 + u_D) v \; dx = \int_\Omega f v \; dx - B(u_D, v) \qquad \text{for all } v \in W_0^{1,2}(\Omega).$$

and therefore $u := \bar{u}_0 + u_D$ is a weak solution to (1).