Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ______________________________

Question	1	2	**Score**
Maximum points	100	100	200
Points			

[100] 1. Consider $\Omega := (-1,1)^3$. Assume that $\{u^n\}_{n=1}^{\infty} \subset \mathcal{C}^{0,1}(\overline{\Omega})$ and that

$$u^n \rightharpoonup 0 \qquad \text{weakly in } W^{1,2}(\Omega).$$

For which $q > 0$ does the following statement hold true?

$$\lim_{n\to\infty} \int_{\partial\Omega} |u|^q \, dS = 0$$

70% Prove in detail the positive result

30% Find counterexample for the remaining case - or at least provide reasonable arguments why it should not be true

Solution:

The claim is true for all $q \in (0,4)$, it is enough to prove it for any $q \in (2,4)$ since then it will clearly be true also for $q \leq 2$. One can do it in general as: The trace operator is continuous to $W^{\frac{1}{2},2}(\partial\Omega)$ and therefore

$$u^n \rightharpoonup 0 \qquad \text{weakly in } W^{1/2,2}(\partial\Omega)$$

and then one can use the compact embedding of the Sobolev spaces with noninteger derivatives to get ($W^{1/2,2}(\partial\Omega) \hookrightarrow\hookrightarrow L^p(\partial\Omega)$ for all $p \in [1,4)$)

$$u^n \to 0 \qquad \text{strongly in } L^q(\partial\Omega) \quad \forall q < 4.$$

However, you were asked for justification, which would mean the proof of the above mentioned embeddings. Also if we look for a counterexample it is evident that we must look for a function which tends to zero but $\|u^n\|_{L^4(\partial\Omega)} \geq \varepsilon$. Therefore, we jsut prove it in hand just following the proof of the trace theorem.

Our first goal is to show that

$$\lim_{n\to\infty} \int_{(-1,1)^2} |u^n(x_1,x_2,-1)|^q dx_1 \, dx_2 = 0 \quad \forall q \in (2,4). \tag{1}$$

Note that this is one of the parts of the boundary and the rest is done similarly. We just recall the three dimensional compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^p$ for any $p \in [1,6)$ and therefore from the assumption we have that

$$\lim_{n\to\infty} \int_{\Omega} |u^n|^p \, dx = 0 \qquad \forall p \in [1,6). \tag{2}$$

Then using the Lipschitz continuity of u^n, we can write ($q \le 4$)

$$\begin{aligned}
2|u^n(x_1,x_2,-1)|^q &= \int_{-1}^{1} \frac{\partial}{\partial x_3}\left((x_3-1)|u^n(x_1,x_2,x_3)|^q\right)\,dx_3 \\
&= \int_{-1}^{1} |u^n(x_1,x_2,x_3)|^q\,dx_3 \\
&+ q\int_{-1}^{1} (x_3-1)|u^n(x_1,x_2,x_3)|^{q-2}u^n(x_1,x_2,x_3)\frac{\partial u^n(x_1,x_2,x_3)}{\partial x_3}\,dx_3 \\
&\le 8\int_{-1}^{1} |u^n(x_1,x_2,x_3)|^q + |u^n(x_1,x_2,x_3)|^{q-1}|\nabla u^n(x_1,x_2,x_3)|\,dx_3.
\end{aligned}$$

Finally, we integrate the result with respect to x_1 and x_2 to observe ($dx = dx_1\,dx_2\,dx_3$)

$$\begin{aligned}
\int_{(-1,1)^2} |u^n(x_1,x_2,-1)|^q\,dx_1\,dx_2 &\le 4\int_\Omega |u^n|^q + |u^n|^{q-1}|\nabla u^n|\,dx \\
&\le 4(\|u^n\|_q^q + \|u^n\|_{1,2}\|u^n\|_{2(q-1)}^{q-1}),
\end{aligned}$$

where for the second inequality we used the Hölder inequality. Finally, since u^n converges weakly in $W^{1,2}$, it must be a bounded sequence and therefore we have

$$\lim_{n\to\infty}\int_{(-1,1)^2} |u^n(x_1,x_2,-1)|^q\,dx_1\,dx_2 \le C\lim_{n\to\infty}(\|u^n\|_q^q + \|u^n\|_{2(q-1)}^{q-1}).$$

Hence, it follows from (2) that if $q < 6$ and $2(q-1) < 6$ then the above limit vanishes. These to assumptions are equivalent to $q < 4$. which finishes the proof.

Note please that we just followed step by step the proof of the trace theorem. For general domain with Lipschtiz boundary it works as well but then one has to employ partition of unity and flattening of the boundary - so it would be more difficult than this example.

For the counterexample, it is just enough to find a sequence of Lipschitz functions, for which $u^n \rightharpoonup 0$ in $W^{1,2}$ but there exists $h > 0$ such that for all n

$$\int_{\partial\Omega} |u^n|^4 \ge h.$$

For simplicity, we shift the domain and consider $\Omega := (-1,1)^2 \times (0,1)$ (otherwise it is done similarly) For this purpose let us define the sequence u^n by the following formula

$$u^n(x_1,x_2,x_3) := (1-n|x|)_+ n^{\frac{1}{2}},$$

where $|x|$ denotes the standard Euclidian norm in $\mathbb{R}^3$ and $a_+ := \max\{0,a\}$. Our goal is to show that for any $\varphi \in L^2$ and any $i = 1,2,3$ we have

$$\lim_{n\to\infty}\int_\Omega \varphi u^n = 0, \tag{3}$$

$$\lim_{n\to\infty}\int_\Omega \varphi\frac{\partial u^n}{\partial x_i} = 0 \tag{4}$$

and that

$$\int_{(-1,1)^2} |u^n(x_1,x_2,0)|^4 \geq h. \tag{5}$$

To justify first two properties, we simply compute from the definition and use the Hölder inequality to get

$$\left|\int_\Omega \varphi u^n\right| = n^{\frac{1}{2}} \left|\int_{\Omega\cap B_{\frac{1}{n}}} \varphi(1-n|x|)\right| \leq n^{\frac{1}{2}} \int_{\Omega\cap B_{\frac{1}{n}}} |\varphi| \leq \|\varphi\|_2 n^{\frac{1}{2}} |B_{\frac{1}{n}}|^{\frac{1}{2}} \leq C\|\varphi\|_2 n^{-1},$$

$$\begin{aligned}\left|\int_\Omega \varphi \frac{\partial u^n}{\partial x_i}\right| &= n^{\frac{3}{2}} \left|\int_{\Omega\cap B_{\frac{1}{n}}} \varphi\frac{x_i}{|x|}\right| \leq n^{\frac{3}{2}} \int_{\Omega\cap B_{\frac{1}{n}}} |\varphi| \leq n^{\frac{3}{2}} |B_{\frac{1}{n}}|^{\frac{3}{2}} \left(\int_{\Omega\cap B_{\frac{1}{n}}} |\varphi|^2\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\Omega\cap B_{\frac{1}{n}}} |\varphi|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $|B_{\frac{1}{n}}| \to 0$ as $n\to\infty$ and $\varphi \in L^2$, it follows from the above computation that (3)–(4) hold true.

Next, we focus on (5). Using the definition of u^n and the substitution theorem, we have

$$\begin{aligned}\int_{(-1,1)^2} |u^n(x_1,x_2,0)|^4\,dx_1\,dx_2 &= n^2 \int_{x_1^2+x_2^2\leq n^{-2}} \left(1-n\sqrt{x_1^2+x_2^2}\right)^4 dx_1\,dx_2 \\ &= \int_{y_1^2+y_2^2\leq 1} (1-|y|)^4\, dy_1\,dy_2 \geq \pi 2^{-7} > 0,\end{aligned}$$

which finishes the proof of (5).

[100] 2. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set and $T > 0$ be arbitrary. Consider the problem: For given $u_0 : \Omega \to \mathbb{R}$ and given $f : (0,T) \times \Omega \to \mathbb{R}$ find $u : (0,T) \times \Omega \to \mathbb{R}$ solving

$$\begin{aligned} \partial_t u - \operatorname{div}(a(|\nabla u|)\nabla u) &= f && \text{in } (0,T) \times \Omega, \\ u &= 0 && \text{on } (0,T) \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned}$$

Assume that $a : \mathbb{R} \to \mathbb{R}$ is continuous function fulfilling for all $s \geq 0$ the following $0 < c_1 \leq a(s) \leq c_2 < \infty$.

10% Find a proper definition of a weak solution. Check that for given data such a definition is meaningful.

40% Find the weakest assumption on a under which you can prove the existence and uniqueness of a weak solution. Justify it. (Hint: Maybe **monotone** operator theory is a good choice if the half of the semester was about it.)

25% Under which conditions on the function a can you do the following?: Show that if f is T-periodic w.r.t. time variable then there exists unique u_0 for which the solution u is also T-periodic with respect to time. (Hint: Maybe **uniformly monotone** operator does the job)

25% Use the conditions on a from the previous step. Consider still T-periodic f_p and denote the periodic solution from the previous step by u^p. Prove that for any weak solution with the right hand side f_p and arbitrary data u_0 there exists constants $K_1, K_2 > 0$ such that for all $n \in \mathbb{N}$

$$\|u(nT) - u^p(0)\|_2 = K_1 e^{-K_2 n}.$$

Solution:

To simplify the notation, we set $A : \mathbb{R}^d \to \mathbb{R}^d$ as

$$A(\xi) := a(|\xi|)\xi.$$

Next, we define $V := W_0^{1,2}(\Omega)$ and $H := L^2(\Omega)$. Then V, Ha and V^* form a Gelfand triple. Finally, for $u_0 \in H$ and $f \in L^2(0,T;V^*)$, we say that u is a weak solution if

$$u \in L^2(0,T;V), \quad \partial_t u \in L^2(0,T;V^*)$$

and for all $\varphi \in V$ and almost all $t \in (0,T)$ there holds

$$\langle \partial_t u, \varphi \rangle_V + \int_\Omega A(\nabla u) \cdot \nabla \varphi = \langle f, \varphi \rangle_V.$$

In addition, we require that $u(0) = u_0$.

Due to the assumptions on f and $\partial_t u$ the duality pairings are well defined for almost all $t \in (0,T)$. Moreover, since (it follows from assumptions on a)

$$|A(\xi)| = a(|\xi|)|\xi| \leq c_2|\xi|, \qquad A(\xi) \cdot \xi = a(|\xi|)|\xi|^2 \geq c_1|\xi|^2 \tag{6}$$

we see that $|A(\nabla u)| \leq c_2|\nabla u|$ and due to the continuity of A and the assumption that $\nabla u \in L^2(0,T;L^2(\Omega))$, also the integral is well defined for almost all time $t \in (0,T)$.

Finally, because u and $\partial_t u$ belong tothe good duality pairing we know that $u \in \mathcal{C}([0,T];H$ and therefore it makes good sense to say $u(0) = u_0 \in H$.

For the existence theorem, we can proceed by themonotone operator theory. Therefore, we need to check what are the assumptions on a such that the operator A is coercive, has linear growth and is monotone. The coercivity and monotonicity was already obtained in (6). We just focus on monotonicity. Hence, let $\xi_1, \xi_2 \in \mathbb{R}^d$ be arbitrary. Then

$$\begin{aligned}(A(\xi_1) - A(\xi_2)) \cdot (\xi_1 - \xi_2) &= (a(|\xi_1|)\xi_1 - a(|\xi_2|)\xi_2) \cdot (\xi_1 - \xi_2) \\ &= a(|\xi_1|)|\xi_1|^2 + a(|\xi_2|)|\xi_2|^2 - (a(|\xi_1|) + a(|\xi_2|))\xi_1 \cdot \xi_2 \\ &\geq a(|\xi_1|)|\xi_1|^2 + a(|\xi_2|)|\xi_2|^2 - (a(|\xi_1|) + a(|\xi_2|))|\xi_1||\xi_2| \\ &= (a(|\xi_1|)|\xi_1| - a(|\xi_2|)|\xi_2|)(|\xi_1| - |\xi_2|).\end{aligned}$$

Hence, we see that

$$(A(\xi_1) - A(\xi_2)) \cdot (\xi_1 - \xi_2) \geq 0,$$

provided that the function $g: \ s \mapsto a(s)s$ is nondecreasing on $[0, \infty)$. (Moreover, one can show that this is in fact the equivalent to the monotonicity of A).

Therefore, if g is nondecreasing we can follow step by step the lecture to get the existence of a solution. Furthermore, to obtain the uniqueness, let us consider two solutions u_1, u_2. Subtracting the weak formulation for u_2 from the weak formulation for u_1, we get that

$$\langle \partial_t(u_1 - u_2), \varphi \rangle_V + \int_\Omega (A(\nabla u_1) - A(\nabla u_2)) \cdot \nabla\varphi = 0$$

for all $\varphi \in V$ and almost all $t \in (0,T)$. Thus setting, $\varphi := (u_1 - u_2)$ and integrating the result over $t \in (0, \tau)$ we have (we use the integration by parts formula for the duality pairing, which is possible thanks to the good duality pairing)

$$\frac{1}{2}\|u_1(\tau) - u_2(\tau)\|_H^2 + \int_0^\tau \int_\Omega (A(\nabla u_1) - A(\nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) = \frac{1}{2}\|u_1(0) - u_2(0)\|_H^2.$$

Fianlly, due to monotonicity of A, the second term is nonnegative, and if $u_1(0) = u_2(0)$ then the above equality implies that $u_1(t) = u_2(t)$ for all $t \in (0,T)$.

Next, we focus on the existence of a periodic solution. It is enough to prove the existence and uniqueness of $u_0 \in H$ for which the unique weak solution satisfies $u(T) = u_0$. Indeed, once having such solution, we can extend it periodically with respect to time thanks to the periodicity of f. To get the existence and uniqueness of such u_0 we use the Banach fixed point theorem. Defining a mapping $S : H \to H$ as $S\, u_0 \mapsto u(T)$, we want to show that it is a contraction.

Assume now that A is uniformly monotone (we check what it means for a later), i.e., that there exists $c_3 > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^d$ we have

$$(A(\xi_1) - A(\xi_2)) \cdot (\xi_1 - \xi_2) \geq c_3|\xi_1 - \xi_2|^2. \tag{7}$$

Then again repeating the same procedure as for the uniqueness we get the inequality

$$\begin{aligned}0 &= \frac{1}{2}\|u_1 - u_2\|_H^2 + \int_\Omega (A(\nabla u_1) - A(\nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) \\ &\overset{(7)}{\geq} \frac{1}{2}\|u_1 - u_2\|_H^2 + c_3\|\nabla u_1 - \nabla u_2\|_2^2 \\ &\overset{\text{Poincaré}}{\geq} \frac{1}{2}\left(\|u_1 - u_2\|_H^2 + c_3 c_p\|u_1 - u_2\|_H^2\right) = \frac{e^{-c_3 c_p t}}{2}\frac{d}{dt}\left(e^{c_3 c_p t}\|u_1 - u_2\|_H^2\right)\end{aligned}$$

Hence, dividing by $\frac{e^{-c_3 c_p t}}{2}$ and integrating over $(0,T)$ we get

$$\|u_1(T) - u_2(T)\|_H^2 \leq e^{-c_3 c_p T} \|u_1(0) - u_2(0)\|_H^2$$

Since $e^{-c_3 c_p T} < 1$, the mapping S is a contraction, and therefore there exists a unique $u_0 \in H$ such that $u(T) = u_0$. This finishes the existence of periodic solution, provided we check the uniform monotonicity (7) of A. Next, we will solve the last part. Integrating the same inequality over $(0, nT)$, we have that for any two solutions

$$\|u_1(nT) - u_2(nT)\|_H^2 \leq e^{-c_3 c_p nT} \|u_1(0) - u_2(0)\|_H^2. \tag{8}$$

Thus, if we set u_2 being the periodic solution u^p then $u_2(nT) = u^p(nT) = u^p(0)$ and the last estimate directly follows from (8).

Finally, we check under what assumptions on a the operator A is uniformly monotone. Thus, assume that $c_3 \leq c_1$ be arbitrary. Then

$$\begin{aligned}
(A(\xi_1) - A(\xi_2)) \cdot (\xi_1 - \xi_2) &= a(|\xi_1|)|\xi_1|^2 + a(|\xi_2|)|\xi_2|^2 - (a(|\xi_1|) + a(|\xi_2|))\xi_1 \cdot \xi_2 \\
&= \frac{(a(|\xi_1|)|\xi_1| - a(|\xi_2|)|\xi_2|)}{|\xi_1| - |\xi_2|}(|\xi_1| - |\xi_2|)^2 + (a(|\xi_1|) + a(|\xi_2|))(|\xi_1||\xi_2| - \xi_1 \cdot \xi_2) \\
&= \left(\frac{(a(|\xi_1|)|\xi_1| - a(|\xi_2|)|\xi_2|)}{|\xi_1| - |\xi_2|} - \varepsilon \right)(|\xi_1| - |\xi_2|)^2 \\
&\quad + \varepsilon(|\xi_1| - |\xi_2|)^2 + (a(|\xi_1|) + a(|\xi_2|))(|\xi_1||\xi_2| - \xi_1 \cdot \xi_2) \\
&= \left(\frac{(a(|\xi_1|)|\xi_1| - a(|\xi_2|)|\xi_2|)}{|\xi_1| - |\xi_2|} - c_3 \right)(|\xi_1| - |\xi_2|)^2 \\
&\quad + c_3|\xi_1 - \xi_2|^2 + (a(|\xi_1|) + a(|\xi_2|) - 2\varepsilon)(|\xi_1||\xi_2| - \xi_1 \cdot \xi_2) \\
&\geq \left(\frac{(a(|\xi_1|)|\xi_1| - a(|\xi_2|)|\xi_2|)}{|\xi_1| - |\xi_2|} - c_3 \right)(|\xi_1| - |\xi_2|)^2 \\
&\quad + c_3|\xi_1 - \xi_2|^2.
\end{aligned}$$

and we see that (7) follows provided that

$$\inf_{s \neq t} \frac{g(s) - g(t)}{s - t} \geq c_3,$$

which in case that g has derivatives implies that $c_3 \leq g'(s) = a'(s)s + a(s)$.