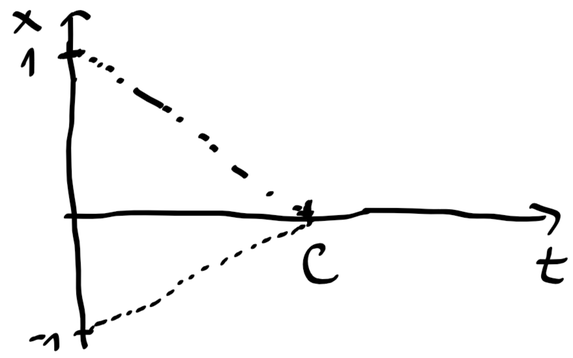
Finite speed of propagation – rigorous for wave eq.

$u \in L^2(0,T; H_0^{1,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap W^{1,2}(0,T; (H_0^{1,2}(\Omega))^*)$

be such that $\partial_t u(0) = u(0) = 0$ in $B_1(0)$ and $\forall w \in H_0^{1,2}(\Omega)$ and a.a $t \in (0,T)$

$$\langle \partial_{tt} u, w \rangle + c^2 \int_\Omega \nabla u \cdot \nabla w = 0$$



Then $u(t,x) = 0$ a.e in $\{(t,x) \; ; \; |x| + \frac{t}{c} < 1\}$

c - speed of PROPAGATION

RIGOROUS PROOF: First $\zeta \Omega$ $c=1$

$\int_0^t$ [.] $\Rightarrow$ $\forall w \in H_0^{1,2}(\Omega)$ and all $t \in (0,T)$

$$\langle \partial_t u(t), w \rangle + \int_\Omega \left(\int_0^t \nabla u(\tau) d\tau \right) \cdot \nabla w = \langle \partial_t u(0), w \rangle$$

Next, Let $g(t,x)$ be Lipschitz and fulfill

A) $g(t,x) = 0$ for all x, $|x| \geq 1$

B) $g(1,x) = 0$ for all $x \in \mathbb{R}^d$

and set $w := ug$ in [ ] (for a.a $t \in (0,T)$ $w(t) \in H_0^{1,2}(\Omega)$)

and $\int_0^1 dt$

$$\int_0^1 \langle \partial_t u, ug \rangle \, dt + \int_0^1 \int_\Omega \left(\int_0^t \nabla u(\tau) d\tau \right) \cdot \nabla (ug) \, dt = \int_0^1 \langle \partial_t u(0), u(t) g(t) \rangle dt$$

① ② ③

③ = 0 ! Indeed we know: $\partial_t u(0,x) = 0$ for $|x| < 1$ (assumption)

$g(t,x) = 0$ for $|x| > 1$ (follows from A))

$\Rightarrow \partial_t u(0,x) u(t,x) g(t,x) \equiv 0$ a.e in $(0,T) \times \Omega$

(1) $\overset{\partial_t m \in L^2}{=} \int_0^1 \int_\Omega \partial_t m \cdot m g = \frac{1}{2} \int_0^1 \int_\Omega \partial_t |m|^2 g \overset{\text{BY PARTS A}}{=} -\frac{1}{2} \int_0^1 \int_\Omega |m|^2 \partial_t g$

$$+ \frac{1}{2} \left[\int_\Omega |m|^2 g \right]_0^1$$

$$= -\frac{1}{2} \int_0^1 \int_\Omega |m|^2 \partial_t g + \frac{1}{2} \int_\Omega |m(1,x)|^2 g(1,x) - |m(0,x)|^2 g(0,x)$$

$g(1,x) = 0$ — ASSUMPTION B; $g(0,x) = 0$ for $|x| \geq 1$; ASSUMPTION A

$$= -\frac{1}{2} \int_0^1 \int_\Omega |m|^2 \partial_t g$$

(2) $= \int_0^1 \int_\Omega \left(\int_0^t \nabla m(\tau)\, d\tau \right) \cdot \left(\nabla m \, g + m \nabla g \right) dt$

$$= \int_0^1 \int_\Omega \frac{1}{2} \partial_t \left| \int_0^t \nabla m(\tau)\, d\tau \right|^2 g + \int_0^1 \int_\Omega \left(\int_0^t \nabla m(\tau)\, d\tau \right) \cdot \nabla g \, m \, dt$$

$$= -\frac{1}{2} \int_0^1 \int_\Omega \left| \int_0^t \nabla m(\tau)\, d\tau \right|^2 \partial_t g + \int_0^1 \int_\Omega \left(\int_0^t \nabla m(\tau)\, d\tau \right) \cdot \nabla g \, m \, dt$$

$$+ \frac{1}{2} \int_\Omega \left| \int_0^1 \nabla m(\tau)\, d\tau \right|^2 g(1,x) - \left| \int_0^0 \nabla m(\tau)\, d\tau \right|^2 g(0,x) \, dx$$

$g(1,x) = 0$; $\int_0^0 = 0$

$$= -\frac{1}{2} \int_0^1 \int_\Omega \left| \int_0^t \nabla m(\tau)\, d\tau \right|^2 \partial_t g + \int_0^1 \int_\Omega \left(\int_0^t \nabla m(\tau)\, d\tau \right) \cdot \nabla g \, m \, dt$$

(1) + (2) = (3)

$$\Leftrightarrow -\frac{1}{2} \int_0^1 \int_\Omega \left(|m|^2 + \left| \int_0^t \nabla m(\tau)\, d\tau \right|^2 \right) \partial_t g = -\int_0^1 \int_\Omega \left(\int_0^t \nabla m(\tau)\, d\tau \right) \cdot \nabla g \, m \, dt$$

$$\overset{\text{YOUNG}}{\leq} \frac{1}{2} \int_0^1 \int_\Omega |\nabla g| \left(\left| \int_0^t \nabla m(\tau)\, d\tau \right|^2 + |m|^2 \right) dt$$

MOTIVATED BY

CHOICE OF g:

$\varepsilon \in (0,1)$: $g(t,x) := \max\{0, (1-\varepsilon-(1-\varepsilon)|x|-t)\}$ — Lipschitz

$\Rightarrow$ $g(1,x) = \max\{0, -\varepsilon-(1-\varepsilon)|x|\} = 0$ — ASSUMPTION B

$0 \leq g(t,x) \leq g(0,x) = \max\{0, (1-\varepsilon)(1-|x|)\} = 0$ for $|x| \geq 1$ — ASSUMPTION A

$\partial_t g = -\chi_{\{t \leq (1-\varepsilon)(1-|x|)\}} =: -\chi_\varepsilon$

$|\nabla g| = (1-\varepsilon) \chi_{\{t \leq (1-\varepsilon)(1-|x|)\}} = \chi_\varepsilon (1-\varepsilon)$

$$\Rightarrow \quad \frac{1}{2}\int_0^1\int_\Omega \left(|u_t|^2+\left|\int_0^t \nabla u(\tau)\,d\tau\right|^2\right)dx \le \frac{(1-\varepsilon)}{2}\int_0^1\int_\Omega \left(|u_t|^2+\left|\int_0^t \nabla u(\tau)\right|^2\right)dx$$

$$\Rightarrow \quad u_t \equiv 0 \quad \text{a.e in } (0,T)\times\Omega$$

$$\Rightarrow \quad u(t,x)=0 \quad \text{for} \quad t \le (1-\varepsilon)(1-|x|)$$

$\varepsilon>0$ arbitrary $\Rightarrow$ $u(t,x)\equiv 0$ for a.e. (t,x) $t+|x|<1$

GENERAL $c\neq 1$

$$\langle \partial_{tt} u, w\rangle + c^2 \int_\Omega \nabla u \cdot \nabla w = 0 \qquad u(0)=\partial_t u(0)=0 \ \text{ in } B_1(0)$$

Define $\tilde{u}(t,x) := u(ct,x)$ $\qquad \tilde{u}(0)=\partial_t \tilde{u}(0)=0$ in $B_1(0)$

$$\Rightarrow \quad \langle \partial_{tt}\tilde{u}, w\rangle + \int_\Omega \nabla\tilde{u}\cdot\nabla w = 0$$

$$\Rightarrow \quad \tilde{u}(t,x)\equiv 0 \quad \text{for} \quad t+|x|<1$$

$$\Rightarrow \quad u(ct,x)\equiv 0 \quad \text{for} \quad t+|x|<1$$

$$\Rightarrow \quad \boxed{u(t,x)\equiv 0 \quad \text{for} \quad \frac{t}{c}+|x|<1}$$