7. cvičení – Řetízkové pravidlo

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Příklady

1. Vypočtěte derivace složených funkcí

   (a) $z = u\sqrt{1+v^2}$, kde $u = e^{2x}$ a $v = e^{-x}$

   Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

   Tedy musíme nejprve spočítat

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \sqrt{1+v^2} \\
\frac{\partial f}{\partial v} &= u\frac{1}{2} \cdot \frac{2v}{\sqrt{1+v^2}} \\
\frac{\partial u}{\partial x} &= 2e^{2x} \\
\frac{\partial v}{\partial x} &= -e^{-x}
\end{aligned}$$

   Dosadíme

$$\begin{aligned}
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} &= \sqrt{1+v^2} \cdot 2e^{2x} + u\frac{1}{2} \cdot \frac{2v}{\sqrt{1+v^2}} \cdot (-e^{-x}) \\
&= \sqrt{1+e^{-2x}} \cdot 2e^{2x} + e^{2x} \cdot \frac{e^{-x}}{\sqrt{1+e^{-2x}}} \cdot (-e^{-x}) \\
&= \frac{2e^{2x}(1+e^{-2x}) - 1}{\sqrt{1+e^{-2x}}} \\
&= \frac{2e^{2x} + 1}{\sqrt{1+e^{-2x}}}
\end{aligned}$$

   (b) $z = uv^2w^3$, kde $u = \sin x$, $v = -\cos x$ a $w = e^x$

   Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

   Tedy musíme nejprve spočítat

$$\begin{aligned}
\frac{\partial f}{\partial u} &= v^2w^3 \\
\frac{\partial f}{\partial v} &= 2uvw^3 \\
\frac{\partial f}{\partial w} &= 3uv^2w^2 \\
\frac{\partial u}{\partial x} &= \cos x \\
\frac{\partial v}{\partial x} &= \sin x \\
\frac{\partial w}{\partial x} &= e^x
\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial x} &= v^2w^3 \cdot \cos x + 2uvw^3 \cdot \sin x + 3uv^2w^2 \cdot e^x \\ &= \cos^2 xe^{3x} \cos x + 2 \sin x(-\cos x)e^{3x} \sin x + 3 \sin x \cos^2 xe^{2x}e^x \\ &= e^3x \cos x(\cos^2 x - 2 \sin^2 x + 3 \sin x \cos x)\end{aligned}$$

(c) $z = \sin u \cos v$, kde $u = (x-y)^2$ a $v = x^2 - y^2$
Řešení: Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}\end{aligned}$$

Tedy musíme nejprve spočítat

$$\begin{aligned}\frac{\partial f}{\partial u} &= \cos u \cos v \\ \frac{\partial f}{\partial v} &= -\sin u \sin v \\ \frac{\partial u}{\partial x} &= 2(x-y) \\ \frac{\partial u}{\partial y} &= -2(x-y) \\ \frac{\partial v}{\partial x} &= 2x \\ \frac{\partial v}{\partial y} &= -2y\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos u \cos v \cdot 2(x-y) - \sin u \sin v \cdot 2x \\ &= \cos(x-y)^2 \cos(x^2-y^2) \cdot 2(x-y) - \sin(x-y)^2 \sin(x^2-y^2) \cdot 2x\end{aligned}$$

a

$$\begin{aligned}\frac{\partial f}{\partial y} &= \cos u \cos v \cdot (-2(x-y)) - \sin u \sin v \cdot (-2y) \\ &= -\cos(x-y)^2 \cos(x^2-y^2) \cdot 2(x-y) + \sin(x-y)^2 \sin(x^2-y^2) \cdot 2y\end{aligned}$$

(d) $w = yz^2 - x^3$, kde $x = e^{r-t}$, $y = \ln(r+2s+3t)$ a $z = \sqrt{rs+t}$
Řešení: Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \\ \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}\end{aligned}$$

Tedy musíme nejprve spočítat

$$\frac{\partial f}{\partial x} = -3x^2 \qquad \frac{\partial f}{\partial y} = z^2 \qquad \frac{\partial f}{\partial z} = 2yz$$
$$\frac{\partial x}{\partial r} = e^{r-t} \qquad \frac{\partial x}{\partial s} = 0 \qquad \frac{\partial x}{\partial t} = -e^{r-t}$$
$$\frac{\partial y}{\partial r} = \frac{1}{r+2s+3t} \qquad \frac{\partial y}{\partial s} = \frac{2}{r+2s+3t} \qquad \frac{\partial y}{\partial t} = \frac{3}{r+2s+3t}$$
$$\frac{\partial z}{\partial r} = \frac{s}{2\sqrt{rs+t}} \qquad \frac{\partial z}{\partial s} = \frac{r}{2\sqrt{rs+t}} \qquad \frac{\partial z}{\partial t} = \frac{1}{2\sqrt{rs+t}}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial r} &= -3x^2 e^{r-t} + z^2 \cdot \frac{1}{r+2s+3t} + 2yz \cdot \frac{s}{2\sqrt{rs+t}} \\ &= -3e^{2(r-t)}e^{r-t} + (rs+t) \cdot \frac{1}{r+2s+3t} + 2\log(r+2s+3t)\sqrt{rs+t} \cdot \frac{s}{2\sqrt{rs+t}} \\ &= -3e^{3(r-t)} + \frac{rs+t}{r+2s+3t} + s\log(r+2s+3t)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial s} &= -3x^2 \cdot 0 + z^2 \cdot \frac{2}{r+2s+3t} + 2yz \cdot \frac{r}{2\sqrt{rs+t}} \\ &= (rs+t) \cdot \frac{2}{r+2s+3t} + 2\log(r+2s+3t)\sqrt{rs+t} \cdot \frac{r}{2\sqrt{rs+t}} \\ &= \frac{2(rs+t)}{r+2s+3t} + r\log(r+2s+3t)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial t} &= -3x^2(-e^{r-t}) + z^2 \cdot \frac{3}{r+2s+3t} + 2yz \cdot \frac{1}{2\sqrt{rs+t}} \\ &= -3e^{2(r-t)}(-e^{r-t}) + (rs+t) \cdot \frac{3}{r+2s+3t} + 2\log(r+2s+3t)\sqrt{rs+t} \cdot \frac{1}{2\sqrt{rs+t}} \\ &= 3e^{3(r-t)} + \frac{3(rs+t)}{r+2s+3t} + \log(r+2s+3t)\end{aligned}$$

2. Spočtěte parciální derivace $g(x,y) = f(x^2+y^2)$.
   Řešení: Uvažujme funkci $f(u)$, kde $u(x,y) = x^2+y^2$.
   Z řetízkového pravidla máme

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot 2x$$
$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} \cdot 2y$$

3. Nechť $f(x,y) = x^2 + 3y^2$. Určete derivace f vzhledem k polárním souřadnicím.
   Polární souřadnice: $x = r\cos\alpha$, $y = r\sin\alpha$.
   Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = 2x \cdot \cos\alpha + 6y \cdot \sin\alpha = 2r\cos\alpha\cos\alpha + 6r\sin\alpha\sin\alpha$$

a

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} = 2x \cdot (-r \sin \alpha) + 6y \cdot r \cos \alpha$$
$$= -2r^2 \cos \alpha \sin \alpha + 6r^2 \sin \alpha \cos \alpha = 4r^2 \cos \alpha \sin \alpha$$

4. Nechť $f(x, y) = e^{-(x^2+y^2)}$. Určete derivace f vzhledem k polárním souřadnicím.
Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = e^{-(x^2+y^2)}(-2x) \cdot \cos \alpha + e^{-(x^2+y^2)}(-2y) \cdot \sin \alpha$$
$$= -e^{-r^2} 2r \cos^2 \alpha - e^{-r^2} 2r \sin^2 \alpha = -2re^{-r^2}$$

a

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} = e^{-(x^2+y^2)}(-2x) \cdot (-r \sin \alpha) + e^{-(x^2+y^2)}(-2y) \cdot r \cos \alpha$$
$$= e^{-r^2} 2r^2 \cos \alpha \sin \alpha - e^{-r^2} 2r^2 \cos \alpha \sin \alpha = 0$$

5. Ukažte, že funkce $F(x, y, z) = \frac{xy}{z} \ln x + xf\left(\frac{y}{x}, \frac{z}{x}\right)$ vyhovuje vztahu $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = F + \frac{xy}{z}$.
Řešení: Uvažujme funkci $f(u, v)$, kde $u(x, y, z) = \frac{y}{x}$ a $v(x, y, z) = \frac{z}{x}$.
Pak dle řetízkového pravidla je

$$\frac{\partial F}{\partial x} = \frac{y}{z} \log x + \frac{xy}{z} \cdot \frac{1}{x} + f(u, v) + x\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$
$$= \frac{y}{z} \log x + \frac{xy}{z} \cdot \frac{1}{x} + f(u, v) + x\left(\frac{\partial f}{\partial u} \cdot -\frac{y}{x^2} + \frac{\partial f}{\partial v} \cdot -\frac{z}{x^2}\right)$$
$$\frac{\partial F}{\partial y} = \frac{x}{z} \log x + x\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}\right)$$
$$= \frac{x}{z} \log x + x\left(\frac{\partial f}{\partial u} \cdot \frac{1}{x} + \frac{\partial f}{\partial v} \cdot 0\right)$$
$$\frac{\partial F}{\partial z} = -\frac{xy}{z^2} \log x + x\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z}\right)$$
$$= -\frac{xy}{z^2} \log x + x\left(\frac{\partial f}{\partial u} \cdot 0 + \frac{\partial f}{\partial v} \cdot \frac{1}{x}\right)$$

Dosadíme do zadaného vztahu:

$$\left(\frac{xy}{z} \log x + \frac{xy}{z} + xf(u, v) - y\frac{\partial f}{\partial u} - z\frac{\partial f}{\partial v}\right) + \left(\frac{yx}{z} \log x + y\frac{\partial f}{\partial u}\right) + \left(-\frac{xy}{z} \log x + z\frac{\partial f}{\partial v}\right)$$
$$= \frac{xy}{z} \log x + \frac{xy}{z} + xf(u, v) = F + \frac{xy}{z}$$

6. Nechť $g(x, y) = f(x + y, x - y)$, spočtěte $\frac{\partial^2 g}{\partial x \partial y}$ v bodě (a, b).
Řešení: Uvažujme funkci $f(u, v)$, kde $u(x, y) = x + y$, $v(x, y) = x - y$.

Z řetízkového pravidla máme

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} \cdot 1 + \frac{\partial f}{\partial v} \cdot (-1).$$

Pak

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \right)$$

Označíme-li funkce $h(u,v) = \frac{\partial f}{\partial u}$ a $j(u,v) = \frac{\partial f}{\partial v}$, máme z řetízkového pravidla

$$\begin{aligned}
\frac{\partial^2 g}{\partial x \partial y} &= \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial x} - \left(\frac{\partial j}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial j}{\partial v} \cdot \frac{\partial v}{\partial x} \right) \\
&= \frac{\partial h}{\partial u} \cdot 1 + \frac{\partial h}{\partial v} \cdot 1 - \left(\frac{\partial j}{\partial u} \cdot 1 + \frac{\partial j}{\partial v} \cdot 1 \right) \\
&= \frac{\partial^2 f}{\partial u \partial u} + \frac{\partial^2 f}{\partial v \partial u} - \frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial v \partial v}
\end{aligned}$$

Protože jsou splněny předpoklady, tak máme záměnné 2. smíšené parciální derivace. Tedy

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 f}{\partial u \partial u} - \frac{\partial^2 f}{\partial v \partial v}$$

V konkrétním bodě (a,b) pak platí

$$\frac{\partial^2 g}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial u \partial u}(a+b, a-b) - \frac{\partial^2 f}{\partial v \partial v}(a+b, a-b)$$

7. Ukažte, že funkce $F(x,y) = xf(x+y) + yg(x+y)$ vyhovuje rovnici $\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} = 0$.
Řešení: Uvažujme funkce $f(u)$ a $g(u)$, kde $u(x,y) = x + y$. Dále označme $\frac{\partial f}{\partial u} = f'$ a $\frac{\partial g}{\partial u} = g'$. Z řetízkového pravidla je

$$\frac{\partial F}{\partial x} = f + xf' \cdot \frac{\partial u}{\partial x} + yg' \cdot \frac{\partial u}{\partial x} = f + xf' + yg'$$

a

$$\frac{\partial F}{\partial y} = xf' \cdot \frac{\partial u}{\partial y} + g + yg' \cdot \frac{\partial u}{\partial y} = xf' + g + yg'$$

Analogicky pak

$$\begin{aligned}
\frac{\partial^2 F}{\partial x^2} &= f' + f' + xf'' + yg'' \\
\frac{\partial^2 F}{\partial y^2} &= xf'' + g' + g' + yg'' \\
\frac{\partial^2 F}{\partial x \partial y} &= f' + xf'' + g' + yg''
\end{aligned}$$

Po dosazení do rovnice je

$$f' + f' + xf'' + yg'' - 2(f' + xf'' + g' + yg'') + xf'' + g' + g' + yg'' = 0$$

8. Výraz $x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x}$ přetransformujte pro funkci $F(u,v) = f(x,y)$, kde $u = y$ a $v = y/x$.
Řešení: Aplikujeme řetízkové pravidlo

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial F}{\partial v}\left(-\frac{y}{x^2}\right)\\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v}\left(-\frac{y}{x^2}\right)\right) = 2\frac{y}{x^3}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial v \partial v}\cdot\frac{\partial v}{\partial x}\right)\\
&= 2\frac{y}{x^3}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial v \partial v}\cdot -\frac{y}{x^2}\right)\\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v}\left(-\frac{y}{x^2}\right)\right) = \frac{-1}{x^2}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial u \partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 F}{\partial v \partial v}\cdot\frac{\partial v}{\partial y}\right)\\
&= \frac{-1}{x^2}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial u \partial v}\cdot 1 + \frac{\partial^2 F}{\partial v \partial v}\cdot\frac{1}{x}\right)
\end{aligned}
$$

Dosadíme

$$
\begin{aligned}
x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x} &= x\left(2\frac{y}{x^3}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial v \partial v}\cdot -\frac{y}{x^2}\right)\right) + y\left(\frac{-1}{x^2}\frac{\partial F}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 F}{\partial u \partial v}\cdot 1 + \frac{\partial^2 F}{\partial v \partial v}\cdot\frac{1}{x}\right)\right)\\
&= \frac{y}{x^2}\frac{\partial F}{\partial v} - \frac{y^2}{x^2}\frac{\partial^2 F}{\partial u \partial v}
\end{aligned}
$$

Z předpisů pro u a v navíc máme $x = \frac{u}{v}$, $y = u$.
Dohromady

$$
\frac{y}{x^2}\frac{\partial F}{\partial v} - \frac{y^2}{x^2}\frac{\partial^2 F}{\partial u \partial v} = \frac{v^2}{u}\frac{\partial F}{\partial v} - v^2\frac{\partial^2 F}{\partial u \partial v}
$$

9. Nechť $F = (F_1, F_2) : \mathbb{R}^3 \to \mathbb{R}^2$ je zobrazení definované předpisem

$$F(x,y,z) = ((x+1)^2(y+1)(z+2), \sin x \cos(2y+z)).$$

Zobrazení $G : \mathbb{R}^2 \to \mathbb{R}^3$ má v bodě $[2,0]$ derivaci representovanou maticí

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(a) Dokažte, že v bodě $[0,0,0]$ existuje derivace zobrazení $G \circ F$ a spočtěte její representující matici.
(b) Spočtěte derivaci funkce F_1 v bodě $[0,0,0]$ podle vektoru $(1,2,0)$.

Řešení:

(a) Nejdříve najdeme reprezentující matici funkce F. Pro $(x,y,z) \in \mathbb{R}^3$ je

$$
\begin{aligned}
\frac{\partial F_1}{\partial x} &= 2(x+1)(y+1)(z+2) & \frac{\partial F_1}{\partial y} &= (x+1)^2(z+2) & \frac{\partial F_1}{\partial z} &= (x+1)^2(y+1)\\
\frac{\partial F_2}{\partial x} &= \cos x \cos(2y+z) & \frac{\partial F_2}{\partial y} &= \sin x(-2\sin(2y+z)) & \frac{\partial F_2}{\partial z} &= \sin x(-\sin(2y+z))
\end{aligned}
$$

Všechny parciální derivace jsou spojité na $\mathbb{R}^3$, tedy existuje derivace zobrazení F.

V bodě $[0,0,0]$ pak je representována maticí

$$\begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Obě zobrazení mají derivaci, tedy ji má i zobrazení $G(F)$. Matici zobrazení $G(F)$ pak získáme jako

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 4 & 2 \\ -1 & 0 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

(b) Všechny parciální derivace funkce F_1 jsou spojité, tedy existuje derivace F_1. Speciálně pro derivaci funkce F_1 v bodě $[0,0,0]$ máme

$$(4,2,1)$$

Pro derivaci ve směru pak platí

$$D_{(1,2,0)}F_1(0,0,0) = (4,2,1)\cdot(1,2,0) = 4+4+0 = 8.$$

10. Nechť $F = (F_1, F_2, F_3, F_4) : \mathbb{R}^2 \to \mathbb{R}$ je zobrazení definované předpisem

$$F(x,y) = \left(\arctan(x+2y), x+y, (x^2+y^2)\frac{|x|}{1+|y|}, ye^x\right).$$

(a) Ukažte, že v bodě $(-1,1)$ existuje derivace funkce F a spočtěte její reprezentující matici.

(b) Spočtěte $\frac{\partial F_3}{\partial x}(0,0)$, pokud existuje.

Řešení:

(a) Uvažujme $(x,y) \in B((-1,1),\frac{1}{2})$. Pak platí $|x| = -x$, $|y| = y$. Zderivujeme

$$\frac{\partial F_1}{\partial x} = \frac{1}{1+(x+2y)^2} \qquad \frac{\partial F_1}{\partial y} = \frac{2}{1+(x+2y)^2}$$
$$\frac{\partial F_2}{\partial x} = 1 \qquad \frac{\partial F_2}{\partial y} = 1,$$
$$\frac{\partial F_3}{\partial x} = \frac{-3x^2-y^2}{1+y} \qquad \frac{\partial F_3}{\partial y} = \frac{x^3-xy(y+2)}{(1+y)^2}$$
$$\frac{\partial F_4}{\partial x} = e^x y \qquad \frac{\partial F_4}{\partial y} = e^x$$

Všechny parciální derivace jsou v bodě $(-1,1)$ spojité, tedy existuje derivace funkce F. Její representující matice pak je

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 1 \\ -2 & \frac{1}{2} \\ e^{-1} & e^{-1} \end{pmatrix}$$

(b) Spočteme z definice.

$$\frac{\partial F_3}{\partial x}(0,0) = \lim_{t\to 0} \frac{((0+t)^2+0^2)\frac{|0+t|}{1+|0|} - 0}{t} = \lim_{t\to 0} \frac{t^2|t|}{t} = \lim_{t\to 0} t|t| = 0.$$