① $\cos(x+y^2) + \sin(xy) = -1 \qquad y(\pi) = 0$

$$F(x,y) = \cos(x+y^2) + \sin(xy) + 1 \qquad [\pi, 0]$$

(a) $F \in C^1(\mathbb{R}^2)$

(b) $F(\pi, 0) = \cos\pi + \sin(0) + 1 = -1 + 1 = 0$

(c) $\dfrac{\partial F}{\partial y} = -\sin(x+y^2)\cdot 2y + \cos(xy)\cdot x$

$\dfrac{\partial F}{\partial y}(\pi, 0) = 0 + 1\cdot\pi = \pi \neq 0$

Tedy $\exists$ okolí U bodu π a okolí V bodu 0 taková že $U\times V \subset \mathbb{R}^2$

a $\forall x \in U \ \exists!\ y \in V$: $F(x,y) = 0$. Navíc $y \in C^1(U)$.

$\dfrac{\partial F}{\partial x} = -\sin(x+y^2)\cdot 1 + \cos(xy)\cdot y$

$\dfrac{\partial F}{\partial x}(\pi, 0) = 0 + 1\cdot 0 = 0$

$y'(\pi) = -\dfrac{0}{\pi} = \underline{\underline{0}}$

② $-v = \cos(xv) + \arctan(yu) \qquad [\pi, 1, 0, 1]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x\ y\ u\ v$

$u = \sin(x+u) + \sin(y-v)$

$F = (F_1, F_2): \mathbb{R}^{2+2} \to \mathbb{R}^2$

$F_1 = \cos(xv) + \arctan(yu) + v$

$F_2 = \sin(x+u) + \sin(y-v) - u$

(a) $F \in C^1(\mathbb{R}^4)$

(b) $F_1(\pi, 1, 0, 1) = \cos\pi + \arctan 0 + 1 = -1 + 1 = 0$

$F_2(\pi, 1, 0, 1) = \sin\pi + \sin 0 - 0 = 0$

(c) $\dfrac{\partial F_1}{\partial u} = \dfrac{1}{1+y^2u^2}\cdot y \qquad \dfrac{\partial F_1}{\partial v} = -\sin(xv)\cdot x + 1$

$\dfrac{\partial F_2}{\partial u} = \cos(x+u) - 1 \qquad \dfrac{\partial F_2}{\partial v} = \cos(y-v)\cdot(-1)$

v $[\pi, 1, 0, 1]$:

$$\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = -1 + 2 = 1 \neq 0$$

Tedy $\exists$ okolí U body $(\pi,1)$ a okolí V bodu $(0,1)$ tak, že

$\forall (x,y) \in U \quad \exists! (u,v): \quad F(x,y,u,v) = 0.$

$u \in C^1(U), \; v \in C^1(U).$

∂y:
$$-v_y = -\sin(xv)\cdot x v_y + \frac{1}{1+y^2u^2}\cdot(u + y u_y)$$
$$u_y = \cos(x+u)\cdot u_y + \cos(y-v)(1 - v_y)$$

v $[\pi, 1, 0, 1]$

$$-v_y = 0 + 1\cdot(0 + 1\cdot u_y). \quad \to \quad u_y = -v_y$$
$$u_y = -1 u_y + 1\cdot(1 - v_y)$$
$$-v_y = v_y + 1 - v_y$$

$$\boxed{\begin{aligned} v_y &= -1 \\ u_y &= 1 \end{aligned}}$$

(3) $u(0,0) = 1, \quad \nabla u(0,0) = (1,2), \quad F: \mathbb{R}^2 \to \mathbb{R}^3, \quad u \in C^1(\mathbb{R}^2)$

$$F(x,y) = \left(e^{x+y}\, u(x+y, x-y),\ (\sin(x+y) - 1)\, u(2y, -x),\ \operatorname{arctan}(x-y)\, u(y, xy)\right) \quad (x,y) \in \mathbb{R}^2$$

$F'(0,0)$?

- $F \in C^1(\mathbb{R}^2)$ (skládání a nás. C^1 zobr) $\to$ $F'(x,y)$ $\exists$ na $\mathbb{R}^2$
- řetízkové pravidlo: $u(a,b), \quad a(x,y), \; b(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x}$$

$$\frac{\partial F_1}{\partial x} = e^{x+y} 1 \cdot u(x+y, x-y) + e^{x+y}\left(\frac{\partial u}{\partial a}\cdot 1 + \frac{\partial u}{\partial b}\cdot(+1)\right)$$

$$\frac{\partial F_1}{\partial y} = e^{x+y} u(x+y, x-y) + e^{x+y}\left(\frac{\partial u}{\partial a}\cdot 1 + \frac{\partial u}{\partial b}\cdot(-1)\right)$$

$$\frac{\partial F_2}{\partial x} = \cos(x+y)\, u(2y-x) + (\sin(x+y)-1)\cdot \frac{\partial u}{\partial b}\cdot(-1)$$

$$\frac{\partial F_2}{\partial y} = \cos(x+y)\, u(2y, -x) + (\sin(x+y)-1)\, \frac{\partial u}{\partial a}\cdot 2$$

$$\frac{\partial F_3}{\partial x} = \frac{1}{1+(x-y)^2}\cdot 1 \cdot u(y, xy) + \arctan(x-y)\, \frac{\partial u}{\partial b}\cdot y$$

$$\frac{\partial F_3}{\partial y} = \frac{1}{1-(x-y)^2}\cdot(-1)\, u(y, xy) + \arctan(x-y)\left(\frac{\partial u}{\partial a}\cdot 1 + \frac{\partial u}{\partial b}x\right)$$

$\to (0,0)$:

$$\begin{pmatrix} 1\cdot 1 + 1\cdot(1\cdot 1 + 2\cdot 1) & 1\cdot 1 + 1\cdot(1 + 2\cdot(-1)) \\ 1\cdot 1 + -1\cdot 2\cdot -1 & 1\cdot 1 + -1\cdot 1\cdot 2 \\ 1\cdot 1 + 0\cdot 2\cdot 0 & 1\cdot(-1)\cdot 1 + 0(1\cdot 1 + 2\cdot 0) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & 0 \\ 3 & -1 \\ 1 & -1 \end{pmatrix}}}$$