① $f_n = \dfrac{n^2x}{1+n^4x^2}, \quad x\in(0,\infty)$

- fix $x\in(0,\infty)$

$$\lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = \lim \frac{n^2}{n^4}\cdot\frac{x}{\frac{1}{n^2}+x^2} \overset{AL}{=} 0\cdot\frac{x}{0+x^2} = 0$$

$f\equiv 0$

- fix $n\in\mathbb{N}$

$$\sigma_n = \sup_{x\in(0,\infty)}\left\{\left|\frac{n^2x}{1+n^4x^2}\right|\right\}$$

$$f_n' = \frac{n^2(1+n^4x^2)-n^2x\,n^4 2x}{(1+n^4x^2)^2} \qquad n^2+n^6x^2-n^6 2x^2 = 0$$

$$n^2 = x^2n^6$$

$$\frac{1}{n^4} = x^2$$

$$\frac{1}{n^2} = x_0$$

$$f_n\left(\frac{1}{n^2}\right) = \frac{1}{1+1} = \frac{1}{2}$$

Tedy $\sigma_n \geq 1/2 \Rightarrow \lim \sigma_n \neq 0 \Rightarrow f_n \not\rightrightarrows 0$ na $(0,\infty)$.

② $\sum x^2e^{-n^2x^2} \quad x\in[0,\infty)$

- fix $x\in(0,\infty)$ $\qquad x^2\sum\left(e^{-x^2}\right)^{n^2}$ k geometrická řada
$e^{-x^2}<1$ (resp. geom. $\sum$ s n. členy $=0$)

$x=0 \quad \sum 0$ k

tedy $\sum$ bodově k na $[0,\infty)$

- $n\in\mathbb{N}$ $\quad \sigma_n = \sup\left\{x^2e^{-n^2x^2},\ x\in[0,\infty)\right\}$

$$f_n' = 2xe^{-n^2x^2}+x^2e^{-n^2x^2}(-2xn^2) \qquad 1-n^2x^2=0 \quad x^2=\frac{1}{n^2} \quad x=\frac{1}{n}$$

$f_n(1/n) = \dfrac{1}{en^2}$

kraje $f_n(0) = 0$

$\lim\limits_{x\to\infty} x^2e^{-n^2x^2} = 0$ ↑ stačí $l'H$

$$\sigma_n = \frac{1}{en^2}, \quad \sum\frac{1}{en^2}\ k \Rightarrow$$

$$\sum f_n \rightrightarrows \text{ na } [0,\infty)$$

$$F(x) = \sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{(\tan x)^n}{\sqrt{n+1}}}_{f_n}, \quad x \in [0, \tfrac{\pi}{4}]$$

- pro $x \in [0, \frac{\pi}{4})$ je $|\tan x| < 1$

Cauchy: $\lim\limits_{n\to\infty} \sqrt[n]{|f_n|} = \lim\limits_{n\to\infty} \frac{|\tan x|}{\sqrt[n]{(n+1)^{1/2}}} = \frac{|\tan x|}{1} < 1$

Tedy $\sum$ K

pro $x = \frac{\pi}{4}$ je $\tan\frac{\pi}{4} = 1$ $\qquad \sum \frac{(-1)^n}{\sqrt{n+1}}$ K z Leibniz

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

$\frac{1}{\sqrt{n+1}}$ je klesající


$$\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+2}} \qquad \sqrt{n+2} \geq \sqrt{n+1} \checkmark$$

- Abel - Dirichlet

  - $a_n = \frac{(-1)^n}{\sqrt{n+1}}$ $\qquad b_n = (\tan x)^n$

  - $\sum a_n \rightrightarrows$ na $[0, \frac{\pi}{4}]$

(z Leibnize výše konverguje bodově, jelikož neobsahuje $x$, tak i $\rightrightarrows$).

  - $b_n = (\tan x)^n$ $\qquad |b_n| \leq 1$ $\qquad (x \in [0, \frac{\pi}{4}])$

    tedy $b_n$ je stejně omezená

  - fix $x \in [0, \frac{\pi}{4}]$. Pak $\underbrace{(\tan x)^n}_{0 \leq \ \ \leq 1}$ je nerostoucí (geom. posl.)

Tedy z Abele $\sum \rightrightarrows$ na $[0, \frac{\pi}{4}]$.