



21. cvičení – Řetízkové pravidlo

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Příklady

1. Vypočtete derivace složených funkcí

(a) $z = u\sqrt{1+v^2}$, kde $u = e^{2x}$ a $v = e^{-x}$

Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

Tedy musíme nejprve spočítat

$$\begin{aligned}\frac{\partial f}{\partial u} &= \sqrt{1+v^2} \\ \frac{\partial f}{\partial v} &= u \cdot \frac{1}{2} \cdot \frac{2v}{\sqrt{1+v^2}} \\ \frac{\partial u}{\partial x} &= 2e^{2x} \\ \frac{\partial v}{\partial x} &= -e^{-x}\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \sqrt{1+v^2} \cdot 2e^{2x} + u \cdot \frac{1}{2} \cdot \frac{2v}{\sqrt{1+v^2}} \cdot (-e^{-x}) \\ &= \sqrt{1+e^{-2x}} \cdot 2e^{2x} + e^{2x} \cdot \frac{e^{-x}}{\sqrt{1+e^{-2x}}} \cdot (-e^{-x}) \\ &= \frac{2e^{2x}(1+e^{-2x}) - 1}{\sqrt{1+e^{-2x}}} \\ &= \frac{2e^{2x} + 1}{\sqrt{1+e^{-2x}}}\end{aligned}$$

(b) $z = uv^2w^3$, kde $u = \sin x$, $v = -\cos x$ a $w = e^x$

Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

Tedy musíme nejprve spočítat

$$\begin{aligned}\frac{\partial f}{\partial u} &= v^2w^3 \\ \frac{\partial f}{\partial v} &= 2uvw^3 \\ \frac{\partial f}{\partial w} &= 3uv^2w^2 \\ \frac{\partial u}{\partial x} &= \cos x \\ \frac{\partial v}{\partial x} &= \sin x \\ \frac{\partial w}{\partial x} &= e^x\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial x} &= v^2 w^3 \cdot \cos x + 2uvw^3 \cdot \sin x + 3uv^2 w^2 \cdot e^x \\ &= \cos^2 x e^{3x} \cos x + 2 \sin x (-\cos x) e^{3x} \sin x + 3 \sin x \cos^2 x e^{2x} e^x \\ &= e^3 x \cos x (\cos^2 x - 2 \sin^2 x + 3 \sin x \cos x)\end{aligned}$$

(c) $z = \sin u \cos v$, kde $u = (x - y)^2$ a $v = x^2 - y^2$

Řešení: Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}\end{aligned}$$

Tedy musíme nejprve spočítat

$$\begin{aligned}\frac{\partial f}{\partial u} &= \cos u \cos v \\ \frac{\partial f}{\partial v} &= -\sin u \sin v \\ \frac{\partial u}{\partial x} &= 2(x - y) \\ \frac{\partial u}{\partial y} &= -2(x - y) \\ \frac{\partial v}{\partial x} &= 2x \\ \frac{\partial v}{\partial y} &= -2y\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos u \cos v \cdot 2(x - y) - \sin u \sin v \cdot 2x \\ &= \cos(x - y)^2 \cos(x^2 - y^2) \cdot 2(x - y) - \sin(x - y)^2 \sin(x^2 - y^2) \cdot 2x\end{aligned}$$

a

$$\begin{aligned}\frac{\partial f}{\partial y} &= \cos u \cos v \cdot (-2(x - y)) - \sin u \sin v \cdot (-2y) \\ &= -\cos(x - y)^2 \cos(x^2 - y^2) \cdot 2(x - y) + \sin(x - y)^2 \sin(x^2 - y^2) \cdot 2y\end{aligned}$$

(d) $w = yz^2 - x^3$, kde $x = e^{r-t}$, $y = \ln(r + 2s + 3t)$ a $z = \sqrt{rs + t}$

Řešení: Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \\ \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}\end{aligned}$$

Tedy musíme nejprve spočítat

$$\begin{aligned}\frac{\partial f}{\partial x} &= -3x^2 & \frac{\partial f}{\partial y} &= z^2 & \frac{\partial f}{\partial z} &= 2yz \\ \frac{\partial x}{\partial r} &= e^{r-t} & \frac{\partial x}{\partial s} &= 0 & \frac{\partial x}{\partial t} &= -e^{r-t} \\ \frac{\partial y}{\partial r} &= \frac{1}{r+2s+3t} & \frac{\partial y}{\partial s} &= \frac{2}{r+2s+3t} & \frac{\partial y}{\partial t} &= \frac{3}{r+2s+3t} \\ \frac{\partial z}{\partial r} &= \frac{s}{2\sqrt{rs+t}} & \frac{\partial z}{\partial s} &= \frac{r}{2\sqrt{rs+t}} & \frac{\partial z}{\partial t} &= \frac{1}{2\sqrt{rs+t}}\end{aligned}$$

Dosadíme

$$\begin{aligned}\frac{\partial f}{\partial r} &= -3x^2 e^{r-t} + z^2 \cdot \frac{1}{r+2s+3t} + 2yz \cdot \frac{s}{2\sqrt{rs+t}} \\ &= -3e^{2(r-t)} e^{r-t} + (rs+t) \cdot \frac{1}{r+2s+3t} + 2 \log(r+2s+3t) \sqrt{rs+t} \cdot \frac{s}{2\sqrt{rs+t}} \\ &= -3e^{3(r-t)} + \frac{rs+t}{r+2s+3t} + s \log(r+2s+3t) \\ \frac{\partial f}{\partial s} &= -3x^2 \cdot 0 + z^2 \cdot \frac{2}{r+2s+3t} + 2yz \cdot \frac{r}{2\sqrt{rs+t}} \\ &= (rs+t) \cdot \frac{2}{r+2s+3t} + 2 \log(r+2s+3t) \sqrt{rs+t} \cdot \frac{r}{2\sqrt{rs+t}} \\ &= \frac{2(rs+t)}{r+2s+3t} + r \log(r+2s+3t) \\ \frac{\partial f}{\partial t} &= -3x^2 (-e^{r-t}) + z^2 \cdot \frac{3}{r+2s+3t} + 2yz \cdot \frac{1}{2\sqrt{rs+t}} \\ &= -3e^{2(r-t)} (-e^{r-t}) + (rs+t) \cdot \frac{3}{r+2s+3t} + 2 \log(r+2s+3t) \sqrt{rs+t} \cdot \frac{1}{2\sqrt{rs+t}} \\ &= 3e^{3(r-t)} + \frac{3(rs+t)}{r+2s+3t} + \log(r+2s+3t)\end{aligned}$$

2. Spočítejte parciální derivace $g(x, y) = f(x^2 + y^2)$.

Řešení: Uvažujme funkci $f(u)$, kde $u(x, y) = x^2 + y^2$.

Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot 2x \\ \frac{\partial g}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{\partial f}{\partial u} \cdot 2y\end{aligned}$$

3. Necht $f(x, y) = x^2 + 3y^2$. Určete derivace f vzhledem k polárním souřadnicím.

Polární souřadnice: $x = r \cos \alpha$, $y = r \sin \alpha$.

Řešení: Z řetízkového pravidla máme

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = 2x \cdot \cos \alpha + 6y \cdot \sin \alpha = 2r \cos \alpha \cos \alpha + 6r \sin \alpha \sin \alpha$$

a

$$\begin{aligned}\frac{\partial f}{\partial \alpha} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} = 2x \cdot (-r \sin \alpha) + 6y \cdot r \cos \alpha \\ &= -2r^2 \cos \alpha \sin \alpha + 6r^2 \sin \alpha \cos \alpha = 4r^2 \cos \alpha \sin \alpha\end{aligned}$$

4. Necht $f(x, y) = e^{-(x^2+y^2)}$. Určete derivace f vzhledem k polárním souřadnicím.

Řešení: Z řetízkového pravidla máme

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = e^{-(x^2+y^2)}(-2x) \cdot \cos \alpha + e^{-(x^2+y^2)}(-2y) \cdot \sin \alpha \\ &= -e^{-r^2} 2r \cos^2 \alpha - e^{-r^2} 2r \sin^2 \alpha = -2re^{-r^2}\end{aligned}$$

a

$$\begin{aligned}\frac{\partial f}{\partial \alpha} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} = e^{-(x^2+y^2)}(-2x) \cdot (-r \sin \alpha) + e^{-(x^2+y^2)}(-2y) \cdot r \cos \alpha \\ &= e^{-r^2} 2r^2 \cos \alpha \sin \alpha - e^{-r^2} 2r^2 \cos \alpha \sin \alpha = 0\end{aligned}$$

5. Ukažte, že funkce $F(x, y, z) = \frac{xy}{z} \ln x + xf\left(\frac{y}{x}, \frac{z}{x}\right)$ vyhovuje vztahu $x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = F + \frac{xy}{z}$.

Řešení: Uvažujme funkci $f(u, v)$, kde $u(x, y, z) = \frac{y}{x}$ a $v(x, y, z) = \frac{z}{x}$.

Pak dle řetízkového pravidla je

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{y}{z} \log x + \frac{xy}{z} \cdot \frac{1}{x} + f(u, v) + x \left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \right) \\ &= \frac{y}{z} \log x + \frac{xy}{z} \cdot \frac{1}{x} + f(u, v) + x \left(\frac{\partial f}{\partial u} \cdot -\frac{y}{x^2} + \frac{\partial f}{\partial v} \cdot -\frac{z}{x^2} \right) \\ \frac{\partial F}{\partial y} &= \frac{x}{z} \log x + x \left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \right) \\ &= \frac{x}{z} \log x + x \left(\frac{\partial f}{\partial u} \cdot \frac{1}{x} + \frac{\partial f}{\partial v} \cdot 0 \right) \\ \frac{\partial F}{\partial z} &= -\frac{xy}{z^2} \log x + x \left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} \right) \\ &= -\frac{xy}{z^2} \log x + x \left(\frac{\partial f}{\partial u} \cdot 0 + \frac{\partial f}{\partial v} \cdot \frac{1}{x} \right)\end{aligned}$$

Dosadíme do zadaného vztahu:

$$\begin{aligned}&\left(\frac{xy}{z} \log x + \frac{xy}{z} + xf(u, v) - y \frac{\partial f}{\partial u} - z \frac{\partial f}{\partial v} \right) + \left(\frac{yx}{z} \log x + y \frac{\partial f}{\partial u} \right) + \left(-\frac{xy}{z} \log x + z \frac{\partial f}{\partial v} \right) \\ &= \frac{xy}{z} \log x + \frac{xy}{z} + xf(u, v) = F + \frac{xy}{z}\end{aligned}$$

6. Necht $g(x, y) = f(x + y, x - y)$, spočítejte $\frac{\partial^2 g}{\partial x \partial y}$ v bodě (a, b) .

Řešení: Uvažujme funkci $f(u, v)$, kde $u(x, y) = x + y$, $v(x, y) = x - y$.

Z řetízkového pravidla máme

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} \cdot 1 + \frac{\partial f}{\partial v} \cdot (-1).$$

Pak

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \right)$$

Označíme-li funkce $h(u, v) = \frac{\partial f}{\partial u}$ a $j(u, v) = \frac{\partial f}{\partial v}$, máme z řetízkového pravidla

$$\begin{aligned} \frac{\partial^2 g}{\partial x \partial y} &= \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial x} - \left(\frac{\partial j}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial j}{\partial v} \cdot \frac{\partial v}{\partial x} \right) \\ &= \frac{\partial h}{\partial u} \cdot 1 + \frac{\partial h}{\partial v} \cdot 1 - \left(\frac{\partial j}{\partial u} \cdot 1 + \frac{\partial j}{\partial v} \cdot 1 \right) \\ &= \frac{\partial^2 f}{\partial u \partial u} + \frac{\partial^2 f}{\partial v \partial u} - \frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial v \partial v} \end{aligned}$$

Protože jsou splněny předpoklady, tak máme záměnné 2. smíšené parciální derivace. Tedy

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 f}{\partial u \partial u} - \frac{\partial^2 f}{\partial v \partial v}$$

V konkrétním bodě (a, b) pak platí

$$\frac{\partial^2 g}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial u \partial u}(a + b, a - b) - \frac{\partial^2 f}{\partial v \partial v}(a + b, a - b)$$

7. Ukažte, že funkce $F(x, y) = xf(x+y) + yg(x+y)$ vyhovuje rovnici $\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} = 0$.

Řešení: Uvažujme funkce $f(u)$ a $g(u)$, kde $u(x, y) = x + y$. Dále označme $\frac{\partial f}{\partial u} = f'$ a $\frac{\partial g}{\partial u} = g'$. Z řetízkového pravidla je

$$\frac{\partial F}{\partial x} = f + xf' \cdot \frac{\partial u}{\partial x} + yg' \cdot \frac{\partial u}{\partial x} = f + xf' + yg'$$

a

$$\frac{\partial F}{\partial y} = xf' \cdot \frac{\partial u}{\partial y} + g + yg' \cdot \frac{\partial u}{\partial y} = xf' + g + yg'$$

Analogicky pak

$$\frac{\partial^2 F}{\partial x^2} = f' + f' + xf'' + yg''$$

$$\frac{\partial^2 F}{\partial y^2} = xf'' + g' + g' + yg''$$

$$\frac{\partial^2 F}{\partial x \partial y} = f' + xf'' + g' + yg''$$

Po dosazení do rovnice je

$$\begin{aligned} f' + f' + xf'' + yg'' - 2(f' + xf'' + g' + yg'') + xf'' + g' + g' + yg'' \\ = 0 \end{aligned}$$

8. Výraz $x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y \partial x}$ přetransformujte pro funkci $F(u, v) = f(x, y)$, kde $u = y$ a $v = y/x$.

Řešení: Aplikujeme řetězkové pravidlo

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial F}{\partial v} \left(-\frac{y}{x^2} \right) \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v} \left(-\frac{y}{x^2} \right) \right) = 2 \frac{y}{x^3} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial u \partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial v \partial v} \cdot \frac{\partial v}{\partial x} \right) \\ &= 2 \frac{y}{x^3} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial v \partial v} \cdot -\frac{y}{x^2} \right) \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial v} \left(-\frac{y}{x^2} \right) \right) = \frac{-1}{x^2} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial u \partial v} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 F}{\partial v \partial v} \cdot \frac{\partial v}{\partial y} \right) \\ &= \frac{-1}{x^2} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial u \partial v} \cdot 1 + \frac{\partial^2 F}{\partial v \partial v} \cdot \frac{1}{x} \right)\end{aligned}$$

Dosadíme

$$\begin{aligned}x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y \partial x} &= x \left(2 \frac{y}{x^3} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial v \partial v} \cdot -\frac{y}{x^2} \right) \right) + y \left(\frac{-1}{x^2} \frac{\partial F}{\partial v} - \frac{y}{x^2} \left(\frac{\partial^2 F}{\partial u \partial v} \cdot 1 + \frac{\partial^2 F}{\partial v \partial v} \cdot \frac{1}{x} \right) \right) \\ &= \frac{y}{x^2} \frac{\partial F}{\partial v} - \frac{y^2}{x^2} \frac{\partial^2 F}{\partial u \partial v}\end{aligned}$$

Z předpisů pro u a v navíc máme $x = \frac{u}{v}$, $y = u$.

Dohromady

$$\frac{y}{x^2} \frac{\partial F}{\partial v} - \frac{y^2}{x^2} \frac{\partial^2 F}{\partial u \partial v} = \frac{v^2}{u} \frac{\partial F}{\partial v} - v^2 \frac{\partial^2 F}{\partial u \partial v}$$

9. Necht' $F = (F_1, F_2) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ je zobrazení definované předpisem

$$F(x, y, z) = ((x+1)^2(y+1)(z+2), \sin x \cos(2y+z)).$$

Zobrazení $G : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ má v bodě $[2, 0]$ derivaci representovanou maticí

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- (a) Dokažte, že v bodě $[0, 0, 0]$ existuje derivace zobrazení $G \circ F$ a spočtete její reprezentující matici.
(b) Spočtete derivaci funkce F_1 v bodě $[0, 0, 0]$ podle vektoru $(1, 2, 0)$.

Řešení:

- (a) Nejdříve najdeme reprezentující matici funkce F . Pro $(x, y, z) \in \mathbb{R}^3$ je

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= 2(x+1)(y+1)(z+2) & \frac{\partial F_1}{\partial y} &= (x+1)^2(z+2) & \frac{\partial F_1}{\partial z} &= (x+1)^2(y+1) \\ \frac{\partial F_2}{\partial x} &= \cos x \cos(2y+z) & \frac{\partial F_2}{\partial y} &= \sin x (-2 \sin(2y+z)) & \frac{\partial F_2}{\partial z} &= \sin x (-\sin(2y+z))\end{aligned}$$

Všechny parciální derivace jsou spojité na $\mathbb{R}^3$, tedy existuje derivace zobrazení F .

V bodě $[0, 0, 0]$ pak je representována maticí

$$\begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Obě zobrazení mají derivaci, tedy ji má i zobrazení $G(F)$. Matici zobrazení $G(F)$ pak získáme jako

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 4 & 2 \\ -1 & 0 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

- (b) Všechny parciální derivace funkce F_1 jsou spojité, tedy existuje derivace F_1 . Speciálně pro derivaci funkce F_1 v bodě $[0, 0, 0]$ máme

$$(4, 2, 1)$$

Pro derivaci ve směru pak platí

$$D_{(1,2,0)}F_1(0, 0, 0) = (4, 2, 1) \cdot (1, 2, 0) = 4 + 4 + 0 = 8.$$

10. Necht $F = (F_1, F_2, F_3, F_4) : \mathbb{R}^2 \rightarrow \mathbb{R}$ je zobrazení definované předpisem

$$F(x, y) = \left(\arctan(x + 2y), x + y, (x^2 + y^2) \frac{|x|}{1 + |y|}, ye^x \right).$$

- (a) Ukažte, že v bodě $(-1, 1)$ existuje derivace funkce F a spočítejte její reprezentující matici.
(b) Spočítejte $\frac{\partial F_3}{\partial x}(0, 0)$, pokud existuje.

Řešení:

- (a) Uvažujme $(x, y) \in B((-1, 1), \frac{1}{2})$. Pak platí $|x| = -x$, $|y| = y$.
Zderivujeme

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \frac{1}{1 + (x + 2y)^2} & \frac{\partial F_1}{\partial y} &= \frac{2}{1 + (x + 2y)^2} \\ \frac{\partial F_2}{\partial x} &= 1 & \frac{\partial F_2}{\partial y} &= 1, \\ \frac{\partial F_3}{\partial x} &= \frac{-3x^2 - y^2}{1 + y} & \frac{\partial F_3}{\partial y} &= \frac{x^3 - xy(y + 2)}{(1 + y)^2} \\ \frac{\partial F_4}{\partial x} &= e^x y & \frac{\partial F_4}{\partial y} &= e^x \end{aligned}$$

Všechny parciální derivace jsou v bodě $(-1, 1)$ spojité, tedy existuje derivace funkce F . Její reprezentující matice pak je

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 1 \\ -2 & \frac{1}{2} \\ e^{-1} & e^{-1} \end{pmatrix}$$

(b) Spočteme z definice.

$$\frac{\partial F_3}{\partial x}(0,0) = \lim_{t \rightarrow 0} \frac{((0+t)^2 + 0^2)^{\frac{|0+t|}{1+|0|}} - 0}{t} = \lim_{t \rightarrow 0} \frac{t^2|t|}{t} = \lim_{t \rightarrow 0} t|t| = 0.$$