13. cvičení – Určitý integrál 2

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Příklady

Spočtěte Newtonovy integrály:

1. $\displaystyle\int_4^\infty \frac{x}{(x-1)(x-2)(x-3)}\,\mathrm{d}x$

   Řešení:

   Rozkladem na parciální zlomky dostaneme

$$\int_4^\infty \frac{x}{(x-1)(x-2)(x-3)}\,\mathrm{d}x = \frac{1}{2}\int_4^\infty \frac{1}{x-1} + \frac{-4}{x-2} + \frac{3}{x-3}\,\mathrm{d}x$$

   Po integraci

$$= \frac{1}{2}\Big[\ln|x-1| - 4\ln|x-2| + 3\ln|x-3|\Big]_4^\infty = \frac{1}{2}\left[\ln\frac{(x-1)(x-3)^3}{(x-2)^4}\right]_4^\infty = 0 - \frac{1}{2}\ln\frac{3\cdot 1^3}{2^4} = \frac{1}{2}\ln\frac{16}{3}$$

2. $\displaystyle\int_{-\infty}^0 \frac{x}{x^3-1}\,\mathrm{d}x$

   Řešení:

$$\int_{-\infty}^0 \frac{x}{x^3-1}\,\mathrm{d}x = \int_{-\infty}^0 \frac{x}{(x-1)(x^2+x+1)}\,\mathrm{d}x$$

   Po rozkladu na parciální zlomky

$$= \frac{1}{3}\int_{-\infty}^0 \frac{1}{x-1} - \frac{x-1}{x^2+x+1}\,\mathrm{d}x = \frac{1}{3}\int_{-\infty}^0 \frac{1}{x-1} - \frac{1}{2}\frac{2x+1-1-2}{x^2+x+1}\,\mathrm{d}x$$
$$= \frac{1}{3}\int_{-\infty}^0 \frac{1}{x-1} - \frac{1}{2}\frac{2x+1}{x^2+x+1} + \frac{1}{2}\frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x$$

   Po integraci

$$\frac{1}{3}\left[\ln|x-1| - \frac{1}{2}\ln|x^2+x+1| + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}\right]_{-\infty}^0$$
$$= \frac{1}{3}\left[\ln\frac{|x-1|}{\sqrt{|x^2+x+1|}} + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}\right]_{-\infty}^0 = 0 + \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6} - \left(0 + \frac{\sqrt{3}}{3}\cdot\frac{-\pi}{2}\right)$$
$$= \frac{2\pi}{9}\sqrt{3}.$$

3. $\displaystyle\int_0^\pi \frac{\sin x}{\cos^2 x + 1}\,\mathrm{d}x$

   Řešení: Zvolíme substituci $t = \cos x$.

$$\int_1^{-1} \frac{-1}{t^2+1}\,\mathrm{d}t = \int_{-1}^1 \frac{1}{t^2+1}\,\mathrm{d}t = [\arctan t]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}-3e^x+3}\,\mathrm{d}x$

Řešení:

Substituce $t = e^x$. Pak

$$\int_0^{\infty} \frac{1}{t^2-3t+3}\,\mathrm{d}t = \int_0^{\infty} \frac{1}{\left(t-\frac{3}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}t = \left[\frac{2}{\sqrt{3}}\arctan\frac{2t-3}{\sqrt{3}}\right]_0^{\infty}$$
$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{3}\right)\right) = \frac{5\pi}{3\sqrt{3}}$$

5. $\displaystyle\int_0^{\pi} \sin^2 x\cos^2 x\,\mathrm{d}x$

Řešení:

Nejprve upravme na

$$\int_0^{\pi} \sin^2 x\cos^2 x\,\mathrm{d}x = \int_0^{\pi} \frac{1}{4}\sin^2(2x)\,\mathrm{d}x$$

Po substituci $t = 2x$ máme

$$\int_0^{2\pi} \frac{1}{8}\sin^2(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{1}{16}(1-\cos(2t))\,\mathrm{d}t = \left[\frac{1}{16}\left(t-\frac{1}{2}\sin(2t)\right)\right]_0^{2\pi} = \frac{\pi}{8}$$

6. $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\cos x-\cos^3 x}\,\mathrm{d}x$

Řešení: Protože jsme na $\left(0, \frac{\pi}{4}\right)$, můžeme psát

$$\int_0^{\frac{\pi}{4}} \sqrt{\cos x-\cos^3 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sqrt{\cos x}\sqrt{1-\cos^2 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sqrt{\cos x}|\sin x|\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{4}} \sqrt{\cos x}\sin x\,\mathrm{d}x$$

Substituce $t = \cos x$:

$$\int_1^{\sqrt{2}/2} -\sqrt{t}\,\mathrm{d}t = \left[\frac{t^{3/2}}{3/2}\right]_{\sqrt{2}/2}^1 = \frac{2}{3}\left(1-\frac{\sqrt[4]{8}}{\sqrt{8}}\right)$$

7. $\displaystyle\int_{-1}^{1} x^2e^{-x}\,\mathrm{d}x$

Řešení:

Aplikujeme per partes

$$\int_{-1}^{1} x^2e^{-x}\,\mathrm{d}x = \left[-x^2e^{-x}\right]_{-1}^1 - \int_{-1}^{1} -2xe^{-x}\,\mathrm{d}x = \left[-x^2e^{-x}\right]_{-1}^1 + \left[-2xe^{-x}\right]_{-1}^1 - \int_{-1}^{1} -2e^{-x}\,\mathrm{d}x$$
$$= \left[-x^2e^{-x}\right]_{-1}^1 + \left[-2xe^{-x}\right]_{-1}^1 + \left[-2e^{-x}\right]_{-1}^1$$
$$= -e^{-1}+e-2e^{-1}-2e-2e^{-1}+2e = e-\frac{5}{e}$$

8. $\displaystyle\int_0^1 \arccos^2 x \,\mathrm{d}x$

Řešení:
Substituce $x = \cos t$, pak $\mathrm{d}x = -\sin t \,\mathrm{d}t$ a

$$\int_0^{\frac{\pi}{2}} t^2 \sin t \,\mathrm{d}t$$

Dvakrát per partes:

$$\int_0^{\frac{\pi}{2}} t^2 \sin t \,\mathrm{d}t = \left[t^2 \cos t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2t \cos t \,\mathrm{d}t = \left[t^2 \cos t\right]_0^{\frac{\pi}{2}} + [2t \sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \sin t \,\mathrm{d}t$$
$$= \left[t^2 \cos t\right]_0^{\frac{\pi}{2}} + [2t \sin t]_0^{\frac{\pi}{2}} + [2 \cos t]_0^{\frac{\pi}{2}} = 0 - 0 + \pi - 0 + 0 - 2 = \pi - 2$$

9. $\displaystyle\int_0^1 x \arcsin x \,\mathrm{d}x$

Řešení: Substituce $x = \sin t,\ \mathrm{d}x = \cos t \,\mathrm{d}t$:

$$\int_0^{\frac{\pi}{2}} t \sin t \cos t \,\mathrm{d}t = \int_0^{\frac{\pi}{2}} t \frac{1}{2} \sin(2t) \,\mathrm{d}t = \frac{1}{4} \int_0^{\frac{\pi}{2}} 2t \sin(2t) \,\mathrm{d}t$$

Substituce $u = 2t,\ \mathrm{d}u = 2 \,\mathrm{d}t$

$$\frac{1}{8} \int_0^{\pi} u \sin u \,\mathrm{d}u$$

Per partes

$$= \frac{1}{8}[-u \cos u]_0^{\pi} + \frac{1}{8} \int_0^{\pi} \cos u \,\mathrm{d}u = \frac{1}{8}[-u \cos u]_0^{\pi} + \frac{1}{8}[\sin u]_0^{\pi} = \frac{1}{8}\pi$$

10. $\displaystyle\int_0^1 x^2 \sqrt{1 - x^2} \,\mathrm{d}x$

Řešení:
Substituce $x = \sin t,\ \mathrm{d}x = \cos t \,\mathrm{d}t$:

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t \,\mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin^2(2t) \,\mathrm{d}t$$

Substituce $u = 2t,\ \mathrm{d}u = 2 \,\mathrm{d}t$:

$$\int_0^{\pi} \frac{1}{8} \sin^2(u) \,\mathrm{d}u = \int_0^{\pi} \frac{1}{16}(1 - \cos(2u)) \,\mathrm{d}u = \left[\frac{1}{16}\left(u - \frac{1}{2}\sin(2u)\right)\right]_0^{\pi}$$
$$= \frac{1}{16}\pi$$

11. $\displaystyle\int_0^1 \sqrt{\frac{x+1}{x}} \,\mathrm{d}x$

Řešení:
Substituce $t = \sqrt{\frac{x+1}{x+0}}$, pak $x = \frac{-1}{1-t^2}$ a $\mathrm{d}x = \frac{-2t}{(t^2-1)^2} \,\mathrm{d}t$:

$$\int_{\sqrt{2}}^{\infty} t \frac{2t}{(t^2 - 1)^2} \,\mathrm{d}t$$

Rozklad na parciální zlomky

$$= \frac{1}{2}\int_{\sqrt{2}}^{\infty} \frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{-1}{t+1} + \frac{1}{(t+1)^2}\,\mathrm{d}t = \frac{1}{2}\left[\ln|t-1| - \ln|t+1| - \frac{1}{t-1} - \frac{1}{t+1}\right]_{\sqrt{2}}^{\infty}$$

$$= \frac{1}{2}\left[\ln\frac{|t-1|}{|t+1|} - \frac{1}{t-1} - \frac{1}{t+1}\right]_{\sqrt{2}}^{\infty} = -\frac{1}{2}\left(\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1}\right)$$

$$= \frac{1}{2}\left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + 2\sqrt{2}\right)$$

Podmínky věty o substituci: $f(x) = \sqrt{\frac{x+1}{x}}$, $\omega(t) = \frac{-1}{1-t^2}$. Interval $(\alpha,\beta) = (\sqrt{2},\infty)$, $(a,b) = (0,1)$. Platí $\omega((\alpha,\beta)) = (a,b)$ a navíc $\omega' = \frac{-2t}{(t^2-1)^2} \neq 0$ na celém (α,β). (Plyne z: obrázku https://www.geogebra.org/calculator/cb3k7uxu)

12. $\displaystyle\int_4^{\infty} \frac{1}{x^2}\sqrt{\frac{x-2}{x-4}}\,\mathrm{d}x$

Řešení:

Substituce: $t = \sqrt{\frac{x-2}{x-4}}$, $x = \frac{-4t^2+2}{1-t^2}$, $\mathrm{d}x = \frac{-4t}{(1-t^2)^2}\,\mathrm{d}t$.

$$\int_1^{\infty} \frac{(1-t^2)^2}{(-4t^2+2)^2} t \frac{4t}{(1-t^2)^2}\,\mathrm{d}t = \int_1^{\infty} \frac{4t^2}{(4t^2-2)^2}\,\mathrm{d}t = \int_1^{\infty} \frac{4t^2}{((2t-\sqrt{2})(2t+\sqrt{2}))^2}\,\mathrm{d}t$$

Rozkladem na parciální zlomky

$$\int_1^{\infty} -\frac{1}{8\left(\sqrt{2}t+1\right)} + \frac{1}{8\left(\sqrt{2}t+1\right)^2} + \frac{1}{8\left(\sqrt{2}t-1\right)} + \frac{1}{8\left(\sqrt{2}t-1\right)^2}\,\mathrm{d}t$$

Po integraci

$$\left[-\frac{1}{8\sqrt{2}}\ln|\sqrt{2}t+1| + \frac{1}{8\sqrt{2}}\ln|\sqrt{2}t-1| - \frac{1}{8\sqrt{2}}\frac{1}{\sqrt{2}t+1} - \frac{1}{8\sqrt{2}}\frac{1}{\sqrt{2}t-1}\right]_1^{\infty}$$

$$= \left[\frac{1}{8\sqrt{2}}\ln\frac{|\sqrt{2}t-1|}{|\sqrt{2}t+1|} - \frac{1}{8\sqrt{2}}\frac{1}{\sqrt{2}t+1} - \frac{1}{8\sqrt{2}}\frac{1}{\sqrt{2}t-1}\right]_1^{\infty}$$

$$= -\frac{1}{8\sqrt{2}}\left(\ln\frac{|\sqrt{2}-1|}{|\sqrt{2}+1|} - \frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1}\right)$$

$$= \frac{1}{8\sqrt{2}}\left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + 2\sqrt{2}\right)$$

Podmínky věty o substituci: $f(x) = \frac{1}{x^2}\sqrt{\frac{x-2}{x-4}}$, $\omega(t) = \frac{-4t^2+2}{1-t^2}$. Interval $(\alpha,\beta) = (1,\infty)$, $(a,b) = (4,\infty)$. Platí $\omega((\alpha,\beta)) = (a,b)$ a navíc $\omega' = \frac{-4t}{(1-t^2)^2} \neq 0$ na celém (α,β). (Plyne z:

$$\omega(t) = \frac{-4t^2+2}{1-t^2} = \frac{-4t^2+4}{1-t^2} + \frac{-2}{1-t^2} = 4 + \frac{-2}{1-t^2},$$

což už lze načrtnout: https://www.geogebra.org/calculator/cb3k7uxu)

13. $\displaystyle\int_0^{4\pi} \frac{1}{\cos x + 2\sin x + 3}\,\mathrm{d}x$

Řešení:

Protože funkce je 2π periodická, můžeme psát

$$\int_0^{4\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x = \int_{-\pi}^{3\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x$$

Což rozepíšeme na

$$\int_{-\pi}^{3\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x + \int_{\pi}^{3\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x$$

Na intervalech $(-\pi,\pi)$ a $(\pi,3\pi)$ pak můžeme substituovat $t = \tan\frac{x}{2}$. Po aplikaci vzorců dostaneme

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{2}{2t^2+4t+4}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{2}{2t^2+4t+4}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{1}{(t+1)^2+1}\,\mathrm{d}t = [\arctan(t+1)]_{-\infty}^{\infty} = \pi\end{aligned}$$

Na intervalu $(\pi,3\pi)$ dostaneme stejný výsledek, celkem tedy máme

$$\int_{-\pi}^{3\pi} \frac{\mathrm{d}x}{\cos x + 2\sin x + 3}\,\mathrm{d}x = 2\pi.$$

Podmínky věty o substituci: $f(t) = \frac{2}{2t^2+4t+4}$, $\omega(x) = \tan\frac{x}{2}$. Interval $(\alpha,\beta) = (-\pi,\pi)$, $(a,b) = (-\infty,\infty)$. Platí $\omega((\alpha,\beta)) = (a,b)$ a navíc $\omega' = \frac{1}{2\cos^2\frac{x}{2}} \neq 0$ na celém (α,β).
Pro interval $(-\pi,3\pi)$ analogicky.

14. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\operatorname{tg} x}$

Řešení:

Substituujeme za $t = \tan x$. Dostaneme

$$\begin{aligned}\int_0^{\infty} \frac{1}{(1+t)(1+t^2)}\,\mathrm{d}t &= \int_0^{\infty} \frac{\frac{1}{2}}{1+t} - \frac{1}{2}\frac{t-1}{(1+t^2)}\,\mathrm{d}t = \int_0^{\infty} \frac{\frac{1}{2}}{1+t} - \frac{1}{4}\frac{2t}{(1+t^2)} + \frac{1}{2}\frac{1}{(1+t^2)}\,\mathrm{d}t \\ &= \left[\frac{1}{2}\ln|1+t| - \frac{1}{4}\ln(t^2+1) + \frac{1}{2}\arctan t\right]_0^{\infty} \\ &= \left[\frac{1}{4}\ln\frac{(1+t)^2}{t^2+1} + \frac{1}{2}\arctan t\right]_0^{\infty} = \frac{\pi}{4}\end{aligned}$$

Podmínky věty o substituci: $f(t) = \frac{1}{(1+t)(1+t^2)}$, $\omega(x) = \tan x$. Interval $(\alpha,\beta) = (0,\frac{\pi}{2})$, $(a,b) = (0,\infty)$. Platí $\omega((\alpha,\beta)) = (a,b)$ a navíc $\omega' = \frac{1}{\cos^2 x} \neq 0$ na celém (α,β).

Zkouškové příklady

doc. Rokyty: https://www2.karlin.mff.cuni.cz/~rokyta/vyuka/index.html
prof. Spurného: https://www2.karlin.mff.cuni.cz/~spurny/pages/ma2.php#

15. $\displaystyle\int_{-\pi}^{\pi} \frac{2+\cos x}{3+\sin x+\cos x}\,\mathrm{d}x$

Řešení: Substituujeme $y = \tan\frac{x}{2}$, pak dostaneme

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{y^2+3}{(y^2+1)(y^2+y+2)}\,\mathrm{d}y &= \int_{-\infty}^{\infty} \frac{y+1}{y^2+y+2} + \frac{-y+1}{y^2+1}\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\cdot\frac{2y+1}{y^2+y+2} + \frac{1}{2}\cdot\frac{1}{y^2+y+2} - \frac{1}{2}\cdot\frac{2y}{y^2+1} + \frac{1}{y^2+1}\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\cdot\frac{2y+1}{y^2+y+2} + \frac{1}{2}\cdot\frac{1}{(y+\frac{1}{2})^2+\frac{7}{4}} - \frac{1}{2}\cdot\frac{2y}{y^2+1} + \frac{1}{y^2+1}\,\mathrm{d}y \\
&= \left[\frac{1}{2}\log(y^2+y+2) + \frac{1}{\sqrt{7}}\arctan\frac{2y+1}{\sqrt{7}} - \frac{1}{2}\log(y^2+1) + \arctan y\right]_{-\infty}^{\infty} \\
&= \left[\frac{1}{2}\log\frac{(y^2+y+2)}{y^2+1} + \frac{1}{\sqrt{7}}\arctan\frac{2y+1}{\sqrt{7}} + \arctan y\right]_{-\infty}^{\infty} \\
&= \pi\left(1+\frac{1}{\sqrt{7}}\right).
\end{aligned}$$

16. $\displaystyle\int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2}\,\mathrm{d}x$

Řešení: Substituujeme $y = \sqrt{2x+1}$. Pak $\mathrm{d}y = \frac{1}{\sqrt{2x+1}}\,\mathrm{d}x$, $x = \frac{1}{2}(y^2-1)$. Dostaneme

$$\begin{aligned}
\int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2}\,\mathrm{d}x &= \int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2}\cdot\frac{\sqrt{2x+1}}{\sqrt{2x+1}}\,\mathrm{d}x \to \int_1^{\sqrt{3}} \frac{4y^2}{(y^2+3)^2}\,\mathrm{d}y \\
&= \int_1^{\sqrt{3}} \frac{4}{y^2+3} - \frac{12}{(y^2+3)^2}\,\mathrm{d}y = \left[\frac{4}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}} - \frac{2y}{y^2+3} - \frac{2}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}}\right]_1^{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{4} - \frac{\sqrt{3}}{3} - \left(\frac{2}{\sqrt{3}}\frac{\pi}{6} - \frac{1}{2}\right) = \frac{1}{\sqrt{3}}\frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{3}
\end{aligned}$$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{3x}}{(e^x+2)^2(e^x+1)^2}\,\mathrm{d}x$

Řešení: Substituce $y = e^x$, $\mathrm{d}y = e^x\,\mathrm{d}x$. Pak

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{2x}\cdot e^x}{(e^x+2)^2(e^x+1)^2}\,\mathrm{d}x &\to \int_0^{\infty} \frac{y^2}{(y+2)^2(y+1)^2}\,\mathrm{d}y \\
&= \int_0^{\infty} \frac{4}{(y+2)^2} + \frac{4}{y+2} + \frac{1}{(y+1)^2} - \frac{4}{y+1}\,\mathrm{d}y \\
&= \left[\frac{-4}{y+2} + 4\log(y+2) - \frac{1}{y+1} - 4\log(y+1)\right]_0^{\infty} \\
&= \left[\frac{-4}{y+2} + 4\log\frac{(y+2)}{y+1} - \frac{1}{y+1}\right]_0^{\infty} = 3 - 4\log 2
\end{aligned}$$

18. $(R)\displaystyle\int_0^1 \frac{\sqrt{x}+2\sqrt[4]{x}}{(\sqrt{x}+1)(\sqrt[4]{x}+1)}\,\mathrm{d}x$

Řešení: Funkce $g(x) = \frac{\sqrt{x}+2\sqrt[4]{x}}{(\sqrt{x}+1)(\sqrt[4]{x}+1)}$ je spojitá na $[0,1]$, tedy existuje Riemannův integrál a rovná se Newtonovu.
Dále tedy počítáme Newtonův integrál. Substituce $y = \sqrt[4]{x}$, $\mathrm{d}y = \frac{1}{4\sqrt[4]{x^3}}\,\mathrm{d}x$. Dostaneme

$$\begin{aligned}\int_0^1 \frac{(y^2+2y)4y^3}{(y^2+1)(y+1)}\,\mathrm{d}y &= 4\int_0^1 y^2+y-2+\frac{y+2}{(y^2+1)(y+1)}\,\mathrm{d}y\\ &= \int_0^1 4y^2+4y-8+\frac{6-2y}{(y^2+1)}+\frac{2}{(y+1)}\,\mathrm{d}y\\ &= \int_0^1 4y^2+4y-8+\frac{-2y}{(y^2+1)}+\frac{6}{(y^2+1)}+\frac{2}{(y+1)}\,\mathrm{d}y\\ &= \left(\frac{4y^3}{3}+2y^2-8y-\log(y^2+1)+6\arctan y+2\log|y+1|\right)_0^1\\ &= -\frac{14}{3}+\log 2+\frac{3\pi}{2}\end{aligned}$$

19. $\displaystyle\int_0^1 \frac{e^x}{e^x+\sqrt{e^{2x}+e^x+1}}\,\mathrm{d}x$

Řešení: Substituce $y = e^x$, $\mathrm{d}y = e^x\,\mathrm{d}x$. Dostaneme

$$\int_1^e \frac{1}{y+\sqrt{y^2+y+1}}\,\mathrm{d}y$$

Aplikujeme Eulerovu substituci

$$\begin{aligned}\sqrt{y^2+y+1} &= s-y\\ y^2+y+1 &= s^2-2sy+y^2\\ y &= \frac{s^2-1}{1+2x}\\ \mathrm{d}y &= \frac{2(s^2+s+1)}{(1+2s)^2}\,\mathrm{d}s\end{aligned}$$

Pak

$$\begin{aligned}\int_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}} \frac{2(s^2+s+1)}{s(1+2s)^2}\,\mathrm{d}s &= \int_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}} \frac{2}{s}-\frac{3}{2s+1}-\frac{3}{(1+2s)^2}\,\mathrm{d}s\\ &= \left[2\log s-\frac{3}{2}\log(2s+1)+\frac{3}{2}\cdot\frac{1}{1+2s}\right]_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}}\\ &= 2\log(\sqrt{3}+1)-\frac{3}{2}\log(2(\sqrt{3}+1)+1)+\frac{3}{2}\cdot\frac{1}{1+2(\sqrt{3}+1)}\\ &\quad-\Big(2\log(e+\sqrt{e^2+e+1})-\frac{3}{2}\log(2(e+\sqrt{e^2+e+1})+1)\\ &\quad+\frac{3}{2}\cdot\frac{1}{1+2(e+\sqrt{e^2+e+1})}\Big)\end{aligned}$$