1

$$\sum \frac{n^2 x}{1+n^4x^2}$$

(a) fix $x \in \mathbb{R}$

$x = 0$ $\quad \sum 0$ ⇝

$x \neq 0$ $\quad$ LSK s $b_n = \frac{1}{n^2}$ $\quad \sum b_n$ ⇝

$$\lim \frac{|f_n|}{b_n} = \lim \frac{n^2|x|}{1+n^4x^2} \cdot \frac{n^2}{1} = \lim \frac{|x|}{\frac{1}{n^4}+x^2} \overset{AL}{=} \frac{|x|}{x^2} \in (0,\infty)$$

tedy $\sum |f_n|$ ⇝, tedy $\sum f_n$ ⇝

$\sum$ ⇝ pro $\forall x \in \mathbb{R}$

(b) f_n jsou spoj na $\mathbb{R}$

fix n, $\sigma_n = \sup \left\{ \frac{n^2|x|}{1+n^4x^2}, \ x \in (0,\infty) \right\}$

$$f_n' = \frac{n^2(1+n^4x^2) - n^2x(n^4 \cdot 2x)}{(1+n^4x^2)^2}$$

$n^2(1+n^4x^2 - 2x^2n^4) = 0$

$1 - n^4x^2 = 0$

$\frac{1}{n^4} = x^2$

$\pm\frac{1}{n^2} = x$

$$f_n\left(\pm\frac{1}{n^2}\right) = \frac{\pm 1}{2}$$

Tedy uvažujme interval (q,∞), kde $q>0$. Od jistého n_0: $\frac{1}{n^2} \notin (q,\infty)$

pak $\lim_{x\to\infty} f_n = \lim \frac{n^2}{\frac{1}{x}+n^4x} \overset{AL}{=} \frac{n^2}{0+\infty} = 0$

tedy $\sigma_n = \frac{n^2q}{1+n^4q^2}$, $\sum \sigma_n$ ⇝ (viz (a))

$\Rightarrow$ $\sum f_n \rightrightarrows$ na (q,∞). Z věty máme $\sum f_n = F$ je spoj na (q,∞). Odtud je spoj i na $(0,\infty)$

② $\sum \frac{(-1)^n}{n} e^{1+\frac{x-1}{n}}$

budeme pracovat na okolí 1: $I = (1-\varepsilon, 1+\delta)$, $\varepsilon, \delta > 0$

(a) $f_n' = \frac{(-1)^n}{n} e^{1+\frac{x-1}{n}} \cdot \frac{1}{n}$

(b) $x_0 = 1$, pak $e \sum \frac{(-1)^n}{n}$ k z Leibnize

(c) fix n,

$$\sigma_n = \sup \left\{ \underbrace{\left| \frac{(-1)^n}{n^2} e^{1+\frac{x-1}{n}} \right|}, x \in I \right\}$$

$$\frac{e^{1+\frac{x-1}{n}}}{n^2} \leq \frac{1}{n^2} e^{1+\frac{1+\delta-1}{n}} = \frac{e^{1+\frac{\delta}{n}}}{n^2} \leq \frac{e^{1+\delta}}{n^2}$$

$\sum e^{1+\delta} \cdot \frac{1}{n^2}$ k odtud $\sum f_n' \rightrightarrows$ na I.

pevné č.

Tedy z věty o záměně sumy a der je $\sum f_n \rightrightarrows$ na I

Tedy f spoj na I (protože f_n jsou spoj na I).

(3)

$$f_n = \frac{(nx+2)^2}{n^2x^2+4}$$

- bod: fix $x \in \mathbb{R} \setminus \{0\}$

$$\lim_{n\to\infty} \frac{(nx+2)^2}{n^2x^2+4} = \lim \frac{n^2}{n^2} \frac{(x+\frac{2}{n})^2}{x^2+\frac{4}{n^2}} \overset{AL}{=} \frac{(x+0)^2}{x^2+0} = 1$$

$x=0 \qquad \lim = \frac{4}{4} = 1$

$f=1$ na $\mathbb{R}$

- stejn. na $[0,2]$

$$\sigma_n = \sup\left\{ \left| \frac{(nx+2)^2}{n^2x^2+4} - 1 \right|, \ x \in [0,2] \right\}$$

$$g_n := f_n - f = \frac{n^2x^2+4 \ +4nx \ - n^2x^2-4}{n^2x^2+4} = \frac{4nx}{n^2x^2+4}$$

$$g_n' = \frac{4n(n^2x^2+4) - 4nx \cdot 2xn^2}{(4+n^2x^2)^2} = \frac{4n^3x^2+16n-8n^3x^2}{(4+n^2x^2)^2}$$

$16n - 4n^3x^2 = 0$

$16n = 4n \cdot n^2x^2$

$\frac{4}{n^2} = x^2$

$\frac{2}{n} = x$

$$g_n\left(\frac{2}{n}\right) = \frac{8}{4+4} = 1$$

tedy $\sigma_n \geq 1$, tedy $f_n \not\rightrightarrows f$ na $[0,2]$

- na $[2,\infty)$

$$\sigma_n = \{ |g_n|, \ x \in [2,\infty) \}$$

sup hledame v krajních bodech

$\lim_{x\to 2} |g_n| = \frac{8n}{4n^2+4}$ tedy $\sigma_n = \frac{8n}{4n^2+4} \to 0$, odtud $f_n \rightrightarrows f$ na $[2,\infty)$

$\lim_{x\to\infty} \frac{4nx}{n^2x^2+4} = 0$

(4)

$$F(x) = \sum \underbrace{\frac{e^{n \sin x}}{n^2}}_{f_n}$$

(a) pro jaká x $\sum$ konverguje?

(b) max intervaly, kde je F spoj.?

(a) fix $x \in \mathbb{R}$, $f_n > 0$, Cauchy

$$\lim_{n\to\infty} \sqrt[n]{f_n} = \lim \frac{e^{\sin x}}{\sqrt[n]{n^2}} = \frac{e^{\sin x}}{1}$$

D pro $e^{\sin x} > 1$ $\quad \sin x > 0$ $\quad x \in (0, \pi) + 2k\pi$

K $\quad e^{\sin x} < 1$ $\quad \sin x < 0$ $\quad x \in (\pi, 2\pi) + 2k\pi$

$e^{\sin x} = 1$ máme $\sum \frac{1}{n^2}$, tedy K., $\sin x = 0$ $\quad x = 0 + k\pi$

Dohromady $\sum K \iff x \in [\pi, 2\pi] + 2k\pi$

(b) fix n, $k \in \mathbb{Z}$:

$$\sigma_n = \sup \left\{ \left| \frac{e^{n \sin x}}{n^2} \right|, \ x \in [\pi, 2\pi] + 2k\pi \right\}$$

$|f_n| \le \frac{1}{n^2}$ $\qquad \sum \frac{1}{n^2}$ K, tedy z Weierstr. věty

$\sum f_n \rightrightarrows$ na $[\pi, 2\pi] + 2k\pi$

navíc f_n jsou spoj na $[\pi, 2\pi) + 2k\pi$, tedy z věty

je F spoj na $[\pi, 2\pi] + 2k\pi$. (Na jiných intervalech není def.)

5

$$f_n = \frac{1}{n^2}\,\frac{|x+\frac{1}{n^2}| - |x-\frac{1}{n^2}|}{|x+\frac{1}{n^2}| + |x-\frac{1}{n^2}|}$$

- $-\frac{1}{n^2}$ $\quad$ $\frac{1}{n^2}$

$$f_n = \begin{cases} \frac{1}{n^2}\cdot\frac{\cancel{-x}-\frac{1}{n^2}-(\cancel{-x}+\frac{1}{n^2})}{-x-\frac{1}{n^2}-x+1/n^2} = \frac{1}{n^2}\cdot\frac{-\frac{2}{n^2}}{-2x} = \frac{1}{n^4x} & x\in(-\infty,-\frac{1}{n^2}] \\ \frac{1}{n^2}\cdot\frac{x+\frac{1}{n^2}-(-x+\frac{1}{n^2})}{x+\frac{1}{n^2}+(-x+\frac{1}{n^2})} = \frac{1}{n^2}\cdot\frac{2x}{\frac{2}{n^2}} = x & x\in(-\frac{1}{n^2},\frac{1}{n^2}) \\ \frac{1}{n^2}\cdot\frac{x+\frac{1}{n^2}-x+\frac{1}{n^2}}{x+\frac{1}{n^2}+x-\frac{1}{n^2}} = \frac{1}{n^2}\cdot\frac{\frac{2}{n^2}}{2x} = \frac{1}{n^4x} & x\in[\frac{1}{n^2},\infty) \end{cases}$$

- bod. konvergence

$x=0$: $\sum \frac{0}{2}$ k

$\sum \frac{1}{n^4x}$ k $\forall x\neq 0$

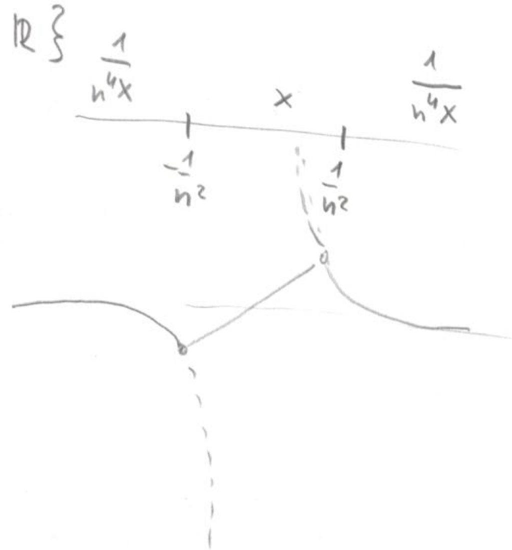
tedy $\sum$ bodově konvergují $\forall x\in\mathbb{R}$

(pro pevné $x\neq 0$ časem „vyhraje" předpis $\sum\frac{1}{n^4x}$)

- stejn. konv.

fix n: $\quad \sigma_n = \sup\{|f_n|,\ x\in\mathbb{R}\}$

$\sigma_n = f_n\left(\frac{1}{n^2}\right) = \frac{1}{n^2}$



kraje $\lim\limits_{x\to\pm\infty}\frac{1}{n^4x}=0$

$\sum\frac{1}{n^2}$ k, tedy $\sum f_n \rightrightarrows$ na $\mathbb{R}$

- spojitost v bodě 3

uvažujme interval $I=(2,4)$

Pak $\sum f_n \rightrightarrows$ na I

navíc $f_n = \frac{1}{n^4 x}$ jsou na I spoj $\rightarrow$ f je na I spoj

- derivace, $I = (2, q)$ $q > 3$

$$f_n = \frac{1}{n^4 x} \quad \text{na } I$$

. $f_n' = \frac{-1}{n^4 x^2}$

. $\sum f_n(3) = \sum \frac{1}{3n^4}$ k

. $\sigma_n = \sup \left\{ \left| \frac{-1}{n^4 x^2} \right|, x \in (2,q) \right\}$

$$= \frac{1}{n^4 \cdot 2^2}$$

$\sum \frac{1}{4n^4}$ k $\rightarrow$ $\sum f_n' \rightrightarrows$ a $f' = \sum f_n' = \sum \frac{-1}{n^4 x^2}$

zřejmě $f' < 0$ (součet záp. $\sum$), tedy f je kles. na $(2,q)$ $\forall q > 3$
na $(2,q)$

Tedy f je kles. na $(2,\infty)$.

1) $f_n(x) = \frac{|\sin x|}{n^2(1+nx^2)}$, $g_n(x) = \frac{1}{n^2(1+nx^2)}$, $n \in \mathbb{N}$, $x \in (-\pi, \pi)$.

a) + b) $|f_n(x)| \le \frac{1}{n^2}$, $x \in (-\pi,\pi)$, $n \in \mathbb{N}$, tedy z Weierstrasse

+3 $\sum f_n$ konverguje stejnoměrně na $(-\pi, \pi)$.

c) $g_n'(x) = \frac{1}{n^2} \frac{-2nx}{(1+nx^2)^2} = -\frac{2}{n} \frac{x}{(1+nx^2)^2}$, $x \in (-\pi, \pi)$.

$$g_n''(x) = -\frac{2}{n} \frac{1}{(1+nx^2)^4} \left((1+nx^2)^2 - x\, 2(1+nx^2)\, 2nx \right) =$$

$$= -\frac{2}{n} \frac{1}{(1+nx^2)^3} \left(1 + nx^2 - 4nx^2 \right) = \frac{-2}{n(1+nx^2)^3} (1 - 3nx^2).$$

$$\Rightarrow \left(g_n''(x) = 0 \iff x = \pm \frac{1}{\sqrt{3n}} \right) \Rightarrow \max_{x \in [0,\pi]} |g_n'(x)| = \left| g_n'\left(\frac{1}{\sqrt{3n}}\right) \right| = \frac{2}{n} \frac{\frac{1}{\sqrt{3n}}}{\left(1 + n\frac{1}{3n}\right)^2} = \frac{2}{\sqrt{3}} \frac{1}{n\sqrt{n}} \frac{1}{\left(\frac{4}{3}\right)^2}$$

+8 $|g_n'(0)| = 0$, $|g_n'(\pi)| = \frac{2\pi}{n(1+n\pi^2)^2}$

$\Rightarrow \sum \|g_n'\|_{[-\pi,\pi]} < \infty \Rightarrow \sum g_n' \rightrightarrows$ na $(-\pi, \pi)$

· $\sum g_n(0)$ konverguje

$\Rightarrow (g(x))' = \left(\sum g_n(x)\right)' = \sum g_n'(x) = \sum_1^\infty -\frac{2}{n} \frac{x}{(1+nx^2)^2}$, $x \in (-\pi, \pi)$.

d) g, f, g' spojité díky stejnoměrné konvergenci, tedy:

· $f_+'(0) = \lim_{x \to 0_+} f'(x) = \lim_{x \to 0_+} (\sin x\, g(x))' = \lim_{x \to 0_+} (\cos x\, g(x) + \sin x\, g'(x)) =$

+3 $= \cos 0\, g(0) + \sin 0\, g'(0) = g(0) = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$

· $f_-'(0) = -\frac{\pi^2}{6}$ ze symetrie.

(7)

$$F(x) = \sum \frac{e^{-nx^2}}{1+n^2}$$

(a) $f_n > 0$, Cauchy

$$\lim_{n\to\infty} \sqrt[n]{f_n} = \lim_{n\to\infty} \frac{e^{-x^2}}{\sqrt[n]{1+n^2}} = \frac{e^{-x^2}}{1} \in (0,1) \quad \forall x \in \mathbb{R}\setminus\{0\}$$

$$1 \leftarrow \sqrt[n]{n^2} \le \sqrt[n]{1+n^2} \le \sqrt[n]{2n^2} \to 1 \qquad \to \sum f_n \text{ k}$$

$x=0$ $\sum 0$ k

Tedy $\sum$ k $\forall x \in \mathbb{R}$

(b) fix n,

$$M_n = \sup\left\{ \frac{e^{-nx^2}}{1+n^2},\ x \in \mathbb{R} \right\}$$

$$\frac{e^{-nx^2}}{1+n^2} \le \frac{1}{1+n^2} \le \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ k $\Rightarrow$ z Weierstr. krit $\sum f_n \rightrightarrows$ na $\mathbb{R}$

f_n spoj na $\mathbb{R}$. Tedy z věty F je spoj na $\mathbb{R}$.

(c) diferenc.:

- $f_n' = \frac{1}{1+n^2} e^{-nx^2} \cdot (-2nx)$
- $\sum f_n(1)$ k (viz (a))
- $M_n = \sup\left\{ \left| \underbrace{\frac{e^{-nx^2}}{1+n^2}(-2nx)}_{} \right|,\ x \in (q,p) \right\}$ kde $q < 1 < p$

$$g_n := \frac{2nx\, e^{-nx^2}}{1+n^2}$$

$$g_n' = \frac{1}{1+n^2}\left(2n e^{-nx^2} + 2nx e^{-nx^2}(-n2x) \right)$$

$$2n e^{-nx^2}(1 - 2nx^2) = 0$$

$$\frac{1}{2n} = x^2 \qquad x = \frac{1}{\sqrt{2n}}$$

(7)

$$f_n'\left(\frac{1}{\sqrt{2n}}\right) = \frac{e^{-1/2}}{1+n^2} \cdot \frac{-2n}{\sqrt{2n}} = -\underbrace{\frac{\sqrt{2}\sqrt{n}}{1+n^2}}_{\approx \frac{1}{n\sqrt{n}}} e^{-1/2}$$

$\sum \frac{1}{n\sqrt{n}}$ k

krajní body

$\sum \frac{e^{-np^2}}{1+n^2} 2np$ k

$$r_n = \frac{\sqrt{2n}\; e^{-1/2}}{1+n^2}$$

srovnáme s geom $\sum (e^{-p^2})^n$

q analog.

$\sum r_n$ k (LSK s $\frac{1}{n\sqrt{n}}$)

tedy z Věty o zaměně sumy a dce

F je diferencovatelná na $\mathbb{R}$.

1) $f_n(x) = \begin{cases} e^{\sqrt[n]{x}} \cdot \log(1+\sqrt[n]{1}) = e^{\sqrt[n]{x}} \log 2, & x \in (0,1] \\ e^{\sqrt[n]{1}} \cdot \log(1+\sqrt[n]{x}) = e \log(1+\sqrt[n]{x}), & x \in (1,\infty) \end{cases}$

a) $\lim_{n\to\infty} f_n(x) = e \log 2, \quad x \in (0,\infty)$ (+4)

b) $\| f_n(x) - e\log 2 \|_\infty = \sup_{x\in(0,\infty)} | f_n(x) - e \log 2 | \geq$ (+4)

$\geq \overline{\lim_{x\to\infty}} | f_n(x) - e\log 2 | = \infty$, tedy $f_n \not\rightrightarrows$ na $(0,\infty)$

c) At $0 < p < 1 < q < \infty$, $x \in [p,q]$. Pak

$= e^{\sqrt[n]{p \wedge 1}} \log(1+\sqrt[n]{p \vee 1}) \leq f_n(x) \leq e^{\sqrt[n]{q \wedge 1}} \log(1+\sqrt[n]{q \vee 1}) =$

$\geq e^{\sqrt[n]{p}} \log 2 \qquad\qquad = e \log(1+\sqrt[n]{q})$

Tj.: $e^{\sqrt[n]{p}} \log 2 - e \log 2 \leq f_n(x) - e\log 2 \leq e\log(1+\sqrt[n]{q}) - e\log 2$
$x \in [p,q]$

přičemž levá i pravá strana konverguje k 0.

Tedy: $\| f_n(x) - e\log 2 \|_\infty \to 0$ pro $x \in [p,q]$.

Tj. $f_n \overset{loc}{\rightrightarrows}$ na $(0,\infty)$ (+8)

2) $f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n(n+1)} x^{2n}$

$g(z) = \sum_{n=1}^{\infty} \frac{n-1}{n(n+1)} z^n$

a) $x \neq 0 \Rightarrow \frac{n}{(n+1)(n+2)} |x|^{2n+2} \cdot \frac{n(n+1)}{n-1} \frac{1}{|x|^{2n}} = \frac{n^2}{(n+2)(n-1)} |x|^2 \to |x|^2$

$\Rightarrow$ poloměr konvergence je 1, střed 0. (+4)

1) $f_n(x) = \dfrac{e^{x^2/n}-1}{n+x^2}$, $x\in\mathbb{R}$, $n\in\mathbb{N}$.

. bodová konvergence: $x\in\mathbb{R}\setminus\{0\}$, pak

. $f_n(x) = \dfrac{e^{x^2/n}-1}{x^2/n}\,\dfrac{x^2}{n}\,\dfrac{1}{n+x^2} \Rightarrow \dfrac{f_n(x)}{1/n^2} \xrightarrow[n\to\infty]{} 1\cdot x^2\cdot 1\in(0,\infty)$

. $f_n(x)>0 \Rightarrow \sum f_n(x)$ konverguje (+3)

. pro $x=0$ je $f_n(0)=0$, tedy $\sum f_n(x)$ konverguje na $\mathbb{R}$.

. stejnoměrná konvergence na $\mathbb{R}$: $\lim\limits_{x\to\infty} f_n(x) = \infty$, tedy

$f_n(x) \not\rightrightarrows 0$ na $\mathbb{R}$, tedy $\sum f_n \not\rightrightarrows$ na $\mathbb{R}$. (+3)

. lokálně stejnoměrná konvergence: ať $x\in[-A,A]$ pro $A>0$ pevné.

$\dfrac{e^y-1}{y}\xrightarrow[y\to 0]{} 1 \Rightarrow$ existuje $\eta>0$, že $\left|\dfrac{e^y-1}{y}\right|\le 2$ pro $y\in(-\eta,\eta)\setminus\{0\}$.

Ať $n_0\in\mathbb{N}$, že $\dfrac{A^2}{n_0}\le\eta$, pak pro $x\in[-A,A]$ a $n\ge n_0$ platí

$|f_n(x)| = \left|\dfrac{e^{x^2/n}-1}{x^2/n}\right|\dfrac{x^2}{n}\cdot\dfrac{1}{n+x^2} \le 2\cdot\dfrac{A^2}{n}\cdot\dfrac{1}{n} = \dfrac{2A^2}{n^2}$, $x\neq 0$

$= 0 \quad \ldots x=0$

Tedy $|f_n(x)|\le\dfrac{2A^2}{n^2}$ a $\sum\dfrac{2A^2}{n^2}<\infty$, proto z Weierstrasse

plyne $\sum f_n \rightrightarrows$ na $[-A,A]$ (+5)

. derivace: $f_n'(x) = \dfrac{1}{(n+x^2)^2}\Big(e^{x^2/n}\underbrace{\dfrac{2x}{n}(n+x^2)}^{2x+\frac{2x^3}{n}} - (e^{x^2/n}-1)\,2x\Big)$

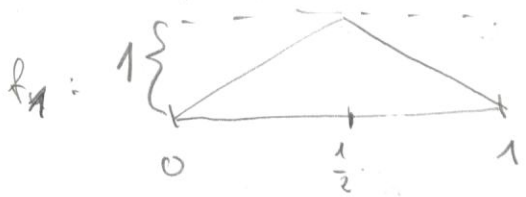
$\Rightarrow$ pro $x\in[-A,A]$ platí $|f_n'(x)|\le\dfrac{1}{n^2}\Big(\underbrace{e^{A^2}(2A+2A^3)+2A(e^{A^2}+1)}_{B}\Big)$

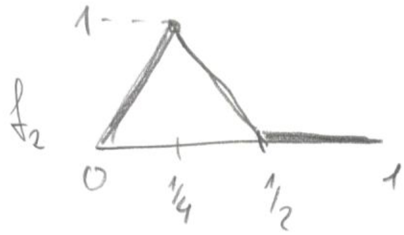
$= \dfrac{B}{n^2}$

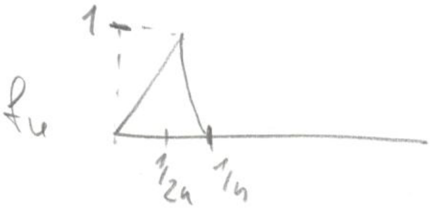
$\overset{\text{Weierstrass}}{\Rightarrow} \sum f_n' \rightrightarrows$ na $(-A,A) \Rightarrow (f(x))' = \sum f_n'(x)$, $x\in(-A,A)$.

. $A>0$ libovolné $\Rightarrow f'(x) = \sum f_n'(x)$, $x\in\mathbb{R}$. (+5)

(10) (a) Neplatí, protipříklad







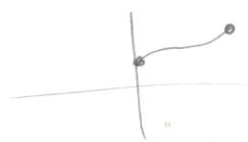
$\rho_n = 1$, tedy $f_n \not\rightrightarrows$

(b) Platí

fix. n,

$$\rho_n = \sup_{x \in [0,1]} \{ | f_n(x) | \} = \max \{ |f_n(0)|, |f_n(1)| \}$$

$\forall f_n$ je rostoucí, tedy sup se musí nabývat v krajních bodech



nebo třeba

pak $\lim_{n \to \infty} \rho_n$

$\leq \lim \; |f_n(0)| + |f_n(1)|$

$= 0 + 0 = 0$

tedy $f_n \rightrightarrows 0$ na $[0,1]$