6. cvičení

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Teorie

Věta 1 (Derivace integrálu podle parametru)**.** Nechť f je **spojitá** funkce na intervalu $[a,b)\times J$, má na celém intervalu **spojitou parciální derivaci** $\frac{\partial f}{\partial y}$, nechť $\int_a^b f(x,y)\,\mathrm{d}x$ **existuje** pro každé $y\in J$. Nechť $\frac{\partial f(x,y)}{\partial y}$ má **integrovatelnou majorantu** na $[a,b)$ vzhledem k $y\in J$. Pak pro každé $y\in J$ máme

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_a^b f(x,y)\,\mathrm{d}x = \int_a^b \frac{\partial f(x,y)}{\mathrm{d}y}\,\mathrm{d}x =$$

Věta 2 (Integrace integrálu dle parametru)**.** Nechť f je **spojitá** funkce na $[a,b)\times[c,d)$ a

1. $f(x,y)$ má **integrovatelnou majorantu** na $[a,b)$ vzhledem k $y\in[c,d)$; $(|f(x,y)|\le g(x))$

   nebo

2. $f(x,y)$ má **integrovatelnou majorantu** na $[c,d)$ vzhledem k $x\in[a,b)$; $(|f(x,y)|\le g(y))$

Pak

$$\int_c^d\left(\int_a^b f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_a^b\left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x.$$

Příklady

1. $$F(\alpha)=\int_0^\infty \frac{1-e^{-\alpha x}}{xe^x}\,\mathrm{d}x \qquad \alpha\in(-1,\infty)$$

2. $$F(\alpha)=\int_0^\infty \frac{1-e^{-\alpha x^2}}{xe^{x^2}}\,\mathrm{d}x \qquad \alpha\in(-1,\infty)$$

3. $$F(\alpha,\beta)=\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,\mathrm{d}x \qquad (\alpha=\beta)\vee(\alpha,\beta>0)$$

4. $$F(\alpha,\beta)=\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x^2}\,\mathrm{d}x \qquad (\alpha=\beta)\vee(\alpha,\beta\ge 0)$$

   Hint: $\int_0^\infty -e^{-\alpha x^2}\,\mathrm{d}x = -\frac{1}{2}\sqrt{\pi/\alpha}$

5. $$F(\alpha)=\int_0^\infty \frac{1-e^{-\alpha x^2}}{x^2e^{x^2}}\,\mathrm{d}x \qquad \alpha\in(-1,\infty)$$

6.

$$F(\alpha,\beta)=\int_0^{\infty}\frac{\operatorname{arctan}\,\alpha x-\operatorname{arctan}\,\beta x}{x}\,\mathrm{d}x \qquad (\alpha=\beta)\vee(\alpha,\beta>0)\vee(\alpha,\beta<0)$$

7.

$$F(\alpha,\beta)=\int_0^{\infty}e^{-\beta x}\frac{\sin\alpha x}{x}\,\mathrm{d}x \qquad \beta\in(0,\infty),\alpha\in\mathbb{R}$$

Hint:dce dle α, $\int_0^{\infty}e^{-\beta x}\cos\alpha x\,\mathrm{d}x=\beta/(\alpha^2+\beta^2)$.

8.

$$F(\alpha)=\int_0^{\pi}\frac{\ln(1+\alpha\cos x)}{\cos x}\,\mathrm{d}x \qquad \alpha\in(-1,1)$$

Hint: $\int_0^{\pi}\frac{\mathrm{d}x}{1+\alpha\cos x}=\frac{\pi}{\sqrt{1-\alpha^2}}$, $\arcsin y'=1/\sqrt{1-y^2}$

9.

$$F(\alpha)=\int_0^{\frac{\pi}{2}}\ln\frac{1+\alpha\cos x}{1-\alpha\cos x}\frac{\mathrm{d}x}{\cos x} \qquad \alpha\in(-1,1)$$

Hint: $\int_0^{\pi/2}\frac{2\,\mathrm{d}x}{1-\alpha^2\cos^2 x}=\frac{\pi}{\sqrt{1-\alpha^2}}$.

10.

$$F(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\operatorname{arctan}\,(\alpha\operatorname{tg}\,x)}{\operatorname{tg}\,x}\,\mathrm{d}x \qquad \alpha\in\mathbb{R}$$

Hint: $\int_0^{\pi/2}\frac{1}{1+\alpha^2\operatorname{tg}^2 x}\,\mathrm{d}x=\frac{\pi}{2}\frac{1}{1+\alpha}$.

11.

$$F(\alpha,\beta)=\int_0^{\frac{\pi}{2}}\ln(\alpha^2\sin^2 x+\beta^2\cos^2 x)\,\mathrm{d}x \qquad (\alpha,\beta)\in\mathbb{R}^2\setminus(0,0)$$

Hint: dce dle α,

$$\int_0^{\pi/2}\frac{2}{\alpha}\frac{\alpha^2\sin^2 x}{\alpha^2\sin^2 x+\beta^2\cos^2 x}\,\mathrm{d}x=\frac{\pi}{\alpha+\beta}$$

Spočtěte integrály

(a)

$$\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x$$

(b)

$$\int_0^{\infty}\frac{\ln(1+a^2x^2)-\ln(1+b^2x^2)}{x^2}\,\mathrm{d}x$$

(c)

$$\int_0^{\infty}\frac{1-e^{-ax^2}}{xe^{x^2}}\,\mathrm{d}x$$

(d)

$$\int_0^{\infty}\frac{\operatorname{arctan}\,ax-\operatorname{arctan}\,bx}{x}\,\mathrm{d}x$$

(e)

$$\int_0^{1}\frac{x^b-x^a}{\ln x}\,\mathrm{d}x$$

(f)

$$\int_0^{\infty}\frac{e^{-ax^2}-e^{-bx^2}}{x}\,\mathrm{d}x$$