Möbius number systems

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NSAC
Novi Sad
August 17–21, 2009

Outline

- Our goal: To use sequences of Möbius transformations to represent points on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
- A Möbius transformation (MT) is any nonconstant function $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$M(z) = \frac{az + b}{cz + d}$$

- We will consider MTs that preserve the upper half-plane.
- These are precisely the MTs with a, b, c, d real and $ad - bc = 1$.

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Classifying Möbius transformations

$$M_0(x) = x/2$$

- Hyperbolic, two fixed points.

$$M_1(x) = x + 1$$

- Parabolic, one fixed point.

$$M_2(x) = -\frac{1}{x+1}$$

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Defining convergence

- A sequence $M_1, M_2, \dots$ represents the number x if $M_n(i) \to x$ for $n \to \infty$.
- Isn't it a bit arbitrary?
- No. This definition is quite natural.
- For example, if $M_1, M_2, \dots$ represents x then $M_n(K) \to \{x\}$ for any K compact lying above the real line.

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Preliminaries from Symbolic dynamics

- Let A be finite alphabet. Let $A^\star$ denote the set of all finite words over A, A^ω the set of all one-sided infinite words.
- $A^\star$ with the operation of concatenation is a monoid.
- Let w_i denote the i-th letter of the word w.
- A set $\Sigma \subset A^\omega$ is a subshift if Σ can be defined by a set of forbidden (finite) factors.
- For $v = v_1 \ldots v_n$ a word, denote by F_v the transformation $F_{v_1} \circ \cdots \circ F_{v_n}$.

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What is a Möbius number system?

Let us have a system of MTs $\{F_a : a \in A\}$. A subshift $\Sigma \subset A^\omega$ is a Möbius number system if:

- For every $w \in \Sigma$, the sequence $\{F_{w_1 \dots w_n}\}_{n=1}^{\infty}$ represents some point $\Phi(w) \in \overline{\mathbb{R}}$.
- The function $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective.

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Getting the idea: Binary system

- Take transformations $F_0(x) = x/2$ and $F_1(x) = (x+1)/2$.
- Take the full shift $\Sigma = \{0, 1\}^\omega$.
- The function Φ maps Σ to an interval on $\overline{\mathbb{R}}$ corresponding to $[0, 1]$.
- Essentially, it is the ordinary binary system; $\Phi(w)$ corresponds to $0.w$.
- Note that this is not a Möbius number system yet, as it is not surjective...
- ...we will fix that soon.

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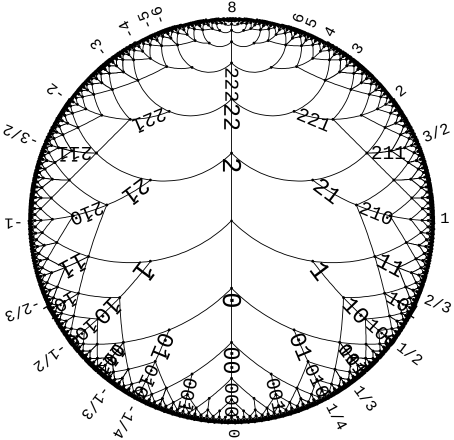
Binary signed system $A = \{\bar{1}, 0, 1, 2\}$



$$
\begin{aligned}
F_{\bar{1}}(x) &= (x-1)/2 \\
F_0(x) &= x/2 \\
F_1(x) &= (x+1)/2 \\
F_2(x) &= 2x
\end{aligned}
$$

Forbidden words:
$20, 02, 12, \bar{1}2, 1\bar{1}, \bar{1}1$

Why forbid words?

- We forbid words to get rid of troublesome combinations.
- In the binary signed system F_0 and F_2 are inverse to each other, so $F_{02} = F_{20} = \mathrm{id}$.
- Forbidding 12 and $\overline{1}2$ keeps twos at the beginning of every word.
- Finally, $1\overline{1}$ and $\overline{1}1$ are forbidden because $\Phi((1\overline{1})^\infty)$ and $\Phi((\overline{1}1)^\infty)$ are not defined.
- We shall see that unregulated concatenation can break *any* Möbius number system.

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Forbidding words is necessary

A *non-erasing substitution* is monoid homomorphism $\rho : A^* \to B^*$ such that $\rho(v)$ is the empty word only for v empty.

Theorem

If Σ is a Möbius number system then $\Sigma \neq \rho(A^\omega)$ for all alphabets A and all non-erasing substitutions ρ.

In particular, for ρ identity we obtain that Σ is never the full shift A^ω.

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Sketch of a proof

- To simplify notation, we consider only the case $\rho(v) = v$.
- We first prove that for every $w \in A^\omega$ and every $x \in \overline{\mathbb{R}}$ it is true that $\lim_{n\to\infty} F_{w_1 w_2 \dots w_n}(x) = \Phi(w)$.
- This is highly suspicious...
- The only way we can obtain such pointwise convergence is when F_v is parabolic (like $x \mapsto x + 1$) for every v nonempty finite word.
- But a simple case consideration shows that then all the F_v have the same fixed point and Φ is a constant map.

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Conclusions

- Sequences of MTs can represent numbers.
- Möbius number systems can emulate more usual means of number representation.
- We can state (and sometimes prove) nontrivial existence conditions such as the one presented . . .
- . . . however, there is a lot of room for improvements.

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Thanks for your attention.