Absorption in idempotent reducts of groups

Alexandr Kazda

Department of Algebra
Charles University, Prague

September 2012

What is absorption?

Definition (Libor Barto, Marcin Kozik)

Let $B \leq A$ be algebras. We say that B absorbs A if there exists a term t in A such that for any $b_1, \ldots, b_n \in B$, $a \in A$ we have:

$$\begin{aligned} t(a, a, a, \ldots, a) &= a \\ t(a, b_2, b_3, \ldots, b_n) &\in B \\ t(b_1, a, b_3, \ldots, b_n) &\in B \\ &\vdots \\ t(b_1, b_2, \ldots, b_{n-1}, a) &\in B \end{aligned}$$

What is absorption?

Definition (Libor Barto, Marcin Kozik)

Let $B \leq A$ be algebras. We say that B absorbs A if there exists a term t in A such that for any $b_1, \ldots, b_n \in B, a \in A$ we have:

$$\begin{aligned} t(a, a, a, \ldots, a) &= a \\ t(a, b_2, b_3, \ldots, b_n) &\in B \\ t(b_1, a, b_3, \ldots, b_n) &\in B \\ &\vdots \\ t(b_1, b_2, \ldots, b_{n-1}, a) &\in B \end{aligned}$$

Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$ then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If A is an algebra with a majority term m then every singleton is an absorbing subalgebra; absorption term is m.
- If A is an algebra then always $A \trianglelefteq A$.
- If A is an abelian group then A has no proper absorbing subalgebra.

Why absorption?

- Absorption is useful for induction-style arguments (see Libor Barto and Marcin Kozik's CSP results).
- If we can absorb, we can often make counterexamples smaller.
- Example: Let P be a connected poset, let $Q \trianglelefteq P$. Then Q is also a connected poset.

Why absorption?

- Absorption is useful for induction-style arguments (see Libor Barto and Marcin Kozik's CSP results).
- If we can absorb, we can often make counterexamples smaller.
- Example: Let P be a connected poset, let $Q \trianglelefteq P$. Then Q is also a connected poset.

Why absorption?

- Absorption is useful for induction-style arguments (see Libor Barto and Marcin Kozik's CSP results).
- If we can absorb, we can often make counterexamples smaller.
- Example: Let P be a connected poset, let $Q \trianglelefteq P$. Then Q is also a connected poset.

Why absorption?

- Absorption is useful for induction-style arguments (see Libor Barto and Marcin Kozik's CSP results).
- If we can absorb, we can often make counterexamples smaller.
- Example: Let P be a connected poset, let $Q \trianglelefteq P$. Then Q is also a connected poset.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Hereditary absorption free algebras

- An algebra A is hereditary absorption free (HAF) if no $B \leq A$ has a proper absorbing subalgebra.
- Example: abelian groups.
- David Stanovský: If A is a finite idempotent solvable algebra then A is hereditary absorption free.
- If A has a Taylor term and is hereditary absorption free then A has a Mal'cev term.
- Consequence (well known in TCT): A abelian with a Taylor term must have a Mal'cev term.

Why groups?

- We want to know what do idempotent HAF algebras look like.
- Groups are the most studied algebras with a Mal'cev term.
- Group terms are something like $x_5^{-2}x_3x_2^{-1}x_1$ – not idempotent. . .
- . . . we have to take the idempotent reduct.

Why groups?

- We want to know what do idempotent HAF algebras look like.
- Groups are the most studied algebras with a Mal'cev term.
- Group terms are something like $x_5^{-2}x_3x_2^{-1}x_1$ – not idempotent. . .
- . . . we have to take the idempotent reduct.

Why groups?

- We want to know what do idempotent HAF algebras look like.
- Groups are the most studied algebras with a Mal'cev term.
- Group terms are something like $x_5^{-2}x_3x_2^{-1}x_1$ – not idempotent…
- … we have to take the idempotent reduct.

Why groups?

- We want to know what do idempotent HAF algebras look like.
- Groups are the most studied algebras with a Mal'cev term.
- Group terms are something like $x_5^{-2} x_3 x_2^{-1} x_1$ – not idempotent...
- ... we have to take the idempotent reduct.

Why groups?

- We want to know what do idempotent HAF algebras look like.
- Groups are the most studied algebras with a Mal'cev term.
- Group terms are something like $x_5^{-2}x_3x_2^{-1}x_1$ – not idempotent...
- ... we have to take the idempotent reduct.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Good news, bad news

- Conjecture: HAF implies solvability for idempotent reducts of groups.
- Good news: Reduct of a group is solvable iff the group is.
- Bad news I: Idempotent reduct of A_5 is HAF, yet not solvable.
- New conjecture: All groups are HAF.
- Bad news II: Subalgebras of a reduct of a group need not be subgroups.

Easy: No absorption in subgroups

- Ignoring bad news makes us happy: If G is a finite group and H is a subgroup of G then H contains no proper absorbing subalgebra.
- To show this, we need to prove that no group contains a proper absorbing subalgebra.
- Let $K \trianglelefteq H$ be such that $K \neq H$. Denote by t the absorption term. We give a proof in the case that K is a subgroup of H.

Easy: No absorption in subgroups

- Ignoring bad news makes us happy: If G is a finite group and H is a subgroup of G then H contains no proper absorbing subalgebra.
- To show this, we need to prove that no group contains a proper absorbing subalgebra.
- Let $K \trianglelefteq H$ be such that $K \neq H$. Denote by t the absorption term. We give a proof in the case that K is a subgroup of H.

Easy: No absorption in subgroups

- Ignoring bad news makes us happy: If G is a finite group and H is a subgroup of G then H contains no proper absorbing subalgebra.
- To show this, we need to prove that no group contains a proper absorbing subalgebra.
- Let $K \trianglelefteq H$ be such that $K \neq H$. Denote by t the absorption term. We give a proof in the case that K is a subgroup of H.

Easy: No absorption in subgroups

- Ignoring bad news makes us happy: If G is a finite group and H is a subgroup of G then H contains no proper absorbing subalgebra.
- To show this, we need to prove that no group contains a proper absorbing subalgebra.
- Let $K \trianglelefteq H$ be such that $K \neq H$. Denote by t the absorption term. We give a proof in the case that K is a subgroup of H.

- Take $h \in H \setminus K$. Then:

$$t(h, e, e, \dots, e) \in K$$
$$t(e, h, e, \dots, e) \in K$$
$$\vdots$$
$$t(e, e, e, \dots, h) \in K$$

- We have a contradiction:

$$t(h, h, \dots, h) = t(h, e, e, \dots, e)t(e, h, e, \dots, e)\dots t(e, e, e, \dots, h)$$

Left side is equal to h, right side belongs into K.

- Take $h \in H \setminus K$. Then:

$$
\begin{gathered}
t(h, e, e, \dots, e) \in K \\
t(e, h, e, \dots, e) \in K \\
\vdots \\
t(e, e, e, \dots, h) \in K
\end{gathered}
$$

- We have a contradiction:

$$t(h, h, \dots, h) = t(h, e, e, \dots, e)t(e, h, e, \dots, e)\dots t(e, e, e, \dots, h)$$

Left side is equal to h, right side belongs into K.

- Take $h \in H \setminus K$. Then:

$$
\begin{gathered}
t(h, e, e, \ldots, e) \in K \\
t(e, h, e, \ldots, e) \in K \\
\vdots \\
t(e, e, e, \ldots, h) \in K
\end{gathered}
$$

- We have a contradiction:

$$t(h, h, \ldots, h) = t(h, e, e, \ldots, e)t(e, h, e, \ldots, e) \ldots t(e, e, e, \ldots, h)$$

Left side is equal to h, right side belongs into K.

Harder: No absorption by singletons

Theorem (E. Aichinger, J. Opršal)

Let A be an idempotent reduct of a group. Then there does not exist $b \in B \leq A$ such that $\{b\} \trianglelefteq B$ (for $|B| > 1$).

Theorem (E. Aichinger, J. Opršal)

Let A be an idempotent reduct of a group. Then there does not exist $b \in B \leq A$ such that $\{b\} \trianglelefteq B$ (for $|B| > 1$).

Open problems

- Are all idempotent reducts of groups HAF?
- What characterizes all HAF idempotent algebras?

Open problems

- Are all idempotent reducts of groups HAF?
- What characterizes all HAF idempotent algebras?

Open problems

- Are all idempotent reducts of groups HAF?
- What characterizes all HAF idempotent algebras?

Thank you for your attention.